Stat 2060 problem set · suggested solutions

1. $X \sim f(x) = \begin{cases} 1.5x^2, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) ① $f(x) \geq 0$

② $\int_{-1}^{1} 1.5x^2\,dx = 1.5\,\frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{2}x^3\Big|_{-1}^{1} = 1$

(b)
(c) $E[X^m] = \int_{-1}^{1} x^m \cdot 1.5x^2\,dx$

$= 1.5\int_{-1}^{1} x^{m+2}\,dx$

$= 1.5 \cdot \frac{1}{m+3}\,x^{m+3}\Big|_{-1}^{1}$

$= \begin{cases} \frac{1.5}{m+3}\cdot 2 = \frac{3}{m+3} & (m \text{ even}) \\ 0 & (m \text{ odd}) \end{cases}$

(d) $EX = 0$, $EX^2 = \frac{3}{2+3} = \frac{3}{5}$

$V(X) = EX^2 - (EX)^2 = \frac{3}{5}$

$SD(X) = \sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5} = 0.7746$

(2)

(e)

$P(X < \eta) = 0.75$

$$\int_{-1}^{\eta} f(x)\,dx = \int_{-1}^{\eta} 1.5x^2\,dx$$

$$= 1.5 \frac{x^3}{3} \Big|_{-1}^{\eta}$$

$$= \frac{\eta^3}{2} - \frac{(-1)}{2} = \frac{\eta^3 + 1}{2} = 0.75$$

$$\eta^3 = 1.5 - 1 = 0.5 \qquad \eta = 0.7937$$

2. There are 20 men 20 women.

(a) P(more than 2 women in committee)

$$= \frac{\binom{20}{3}\binom{20}{1} + \binom{20}{4}}{\binom{40}{4}} =$$

(b) P(no men | 1 minority woman)

$$= \frac{P(\text{no men and 1 minority woman})}{P(\text{1 minority woman})}$$

$$= \frac{\binom{4}{1}\binom{16}{3}}{\binom{36}{3}\binom{4}{1}} = \frac{\binom{16}{3}}{\binom{36}{3}} =$$

(c) $\dfrac{\binom{16}{1}\binom{20}{1}\binom{4}{1}\binom{37}{1}}{\binom{40}{4}}$

3. $n=21 \quad \bar{x}=46.3, \quad s=5$

$H_0: \mu_0 = 43.5$

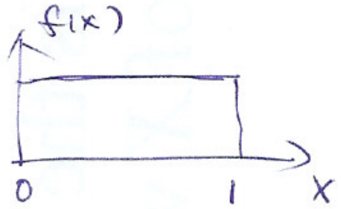
$H_a: \mu_0 > 43.5$

(a) $t = \dfrac{\bar{x}-\mu_0}{s/\sqrt{n}} = \dfrac{46.3-43.5}{5/\sqrt{21}} = 2.566$

(b) p-value $= P(t_{20} > 2.566) < 0.01$
$(=1-0.9908)$

(c) reject H_0 at level $\alpha = 0.05$.

4.



$P(A_1) = P(0.1 < X < 0.25)$
$= 0.15$

$P(A_2) = P(0.15 < X < 0.35) = 0.2$

$P(A_3) = 0.05$

(a) $P(A_1 \cup A_2 \cup A_3) = P(0.1 \le X \le 0.35) + P(0.75 \le X \le 0.8)$
$= 0.25 + 0.05 = 0.3$

(b) $P(A_1 \cap A_2') = P(0.1 \le X \le 0.15) = 0.05$

(4)

5. $p = 0.2$, $n = 100$

(a) mean $= np = 20$

Variance $= npq = 16$

(b) $X \sim Bin(100, 0.2)$

$P(X \leq 15)$

$\approx P(X < 15.5)$ $\quad X \overset{approx.}{\sim} N(20, 16)$

$= P\left(Z < \frac{15.5 - 20}{4}\right)$

$= P(Z < -1.125) = 0.1303$

6. $H_0 : \mu_0 = 9.0$

$H_a : \mu_0 \neq 9.0$

From 95% C.I. is $(6.2, 9.8)$

suppose σ known. this is got from

$\left(\bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$

$$\begin{cases} \bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 6.2 \\ \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 9.8 \\ Z_{\alpha/2} = 1.96 \end{cases} \Rightarrow \begin{cases} \bar{X} = 8 \\ \frac{\sigma}{\sqrt{n}} = 0.9184 \end{cases}$$

Test statistic : $Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{8 - 9}{0.9184} = -1.089$

(5)

$z_{\alpha/2} = z_{0.05} = 1.64$

H_0 can not be rejected.

7. (a) $E[e^X] = e^{-1} \times \frac{1}{4} + e^2 \times \frac{1}{2} + e^5 \times \frac{1}{8} + e^6 \times \frac{1}{8}$

$= \frac{1}{4e} + \frac{e^2}{2} + \frac{e^5}{8} + \frac{e^6}{8}$

(b)

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4} & -1 \le x < 2 \\ \frac{1}{4} + \frac{1}{2} = \frac{3}{4} & 2 \le x < 5 \\ \frac{3}{4} + \frac{1}{8} = \frac{7}{8} & 5 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

8. (a) X = no. of defectives among 10

$X \sim Bin(10, 0.2)$

$P(X \ge 1) = 1 - P(X = 0)$

$= 1 - \binom{10}{0} 0.2^0 \, 0.8^{10}$

$= 0.8926$

(b) $Y = \begin{cases} -50 & \text{if defective. } p = 0.2 \\ 20 & \text{nondefective } p = 0.8 \end{cases}$

X = no. of defectives in 30 calculators.

(b)

$E(\text{profit})$ $\quad X \sim Bin(30, 0.2)$

$EX = 6$.

$= E[-50X + 20(30 - X)]$

$= E[-50X + 600 - 20X]$

$= 600 - 70EX = 600 - 70 \times 6 = 180$

(c) $(0.8)^{10}(0.2)^{1} = 0.0215$

9. (a) $P(A) = 0.7 \quad P(B) = 0.5$

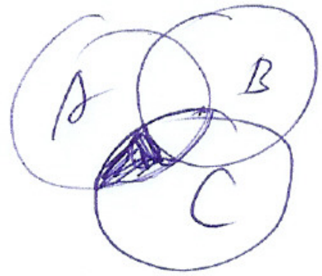
$P(A \cap B) = 0.25$

$P(A' \cup B') = P((A \cap B)')$

$= 1 - P(A \cap B)$

$= 1 - 0.25 = 0.75$

(b) $P(A \cap B^c \cap C)$



$= P(A \cap C) - P(A \cap B \cap C)$

$= 0.25 - 0.2$

$= 0.05$

10. $\boxed{\text{2 gold}}$ $\boxed{15.1g}$ $\boxed{25}$

(a) $\frac{3}{6} = 0.5$

or $P(\text{gold coin chosen})$

$= P(\text{1st drawer}) + P(\text{2nd drawer})P(\text{gold} \mid \text{2nd drawer})$

$= \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3} + \frac{1}{6} = \frac{3}{6} = 0.5$

(b) $P(\text{1st drawer} \mid \text{gold coin})$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

11. $n = 500$. $\hat{p} = \frac{256}{500}$

(a) $H_0: p_0 = \frac{18}{38}$, $H_a: p_0 \neq \frac{18}{38}$.

Test statistic: $z = \dfrac{\hat{p} - \frac{18}{38}}{\sqrt{p_0(1-p_0)/n}}$

$= 1.7159$

p-value $= 0.0862$

(b) $\hat{p} = \frac{256}{500}$, $n=500$, $\alpha = 0.01$

use $$\frac{\hat{p} \pm \frac{z_{\alpha/2}^2}{2n} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + (z_{\alpha/2}^2)/n}.$$

12. $\bar{x} = 26.85$, $s^2 = 2.92$, $n = 10$

(a) $H_0: \mu = 25.65$

$H_a: \mu \neq 25.65$

$$\cancel{t} = \frac{\bar{x} - 25.65}{\sqrt{2.92}/\sqrt{n}} = \frac{26.85 - 25.65}{\sqrt{2.92/10}} = 2.2207$$

$$P\text{-value} = 2(1 - P(t_9 < 2.2207)) = \cancel{0.0264}\ 0.0535$$

at $\alpha = 0.05$, Ho can Not be rejected.

There isn't strong evidence that Gro-food treated plant has ~~the larger~~ different height.

(b) $\bar{x} \pm t_{\alpha/2, 9} \frac{s}{\sqrt{n}}$

$= 26.85 \pm 2.26 \cdot \sqrt{\frac{2.92}{10}}$

95% C.I. is (25.63, 28.07)

13. $X \sim N(10, 9)$

(a) 50th percentile $= 10$

(b) 75th percentile $= 10 + 3 \times Z_{0.25}$

$= 10 + 3 \times 0.6745$

$= 12.02$

14. $X_F \sim N(64, 2.5^2)$

$X_M \sim N(68, 3^2)$

$Y = X_F - X_M \sim N(-4, \underbrace{2.5^2 + 3^2}_{15.25})$

$P(|X_F - X_M| < 4)$

$= P(-4 < Y < 4)$

$= P\left(\frac{-4-(-4)}{\sqrt{15.25}} < Z < \frac{4-(-4)}{\sqrt{15.25}}\right)$

$= P(\quad 0 < Z < 2.0486)$

$= 0.9797 - 0.5 = 0.4797$

15. $X_1, \cdots, X_{12}$ iid $N(\mu, \sigma^2)$ (10)

(a) $\sigma^2 = 0.04$.

$Y = X_1 + \cdots + X_{12} \sim N(12\mu, 12\sigma^2)$

$P(Y > 140) = 0.95$

$P\left(\cancel{Z} > \frac{140 - 12\mu}{\sqrt{12\sigma^2}}\right) = 0.95$

$\frac{140 - 12\mu}{\sqrt{12\sigma^2}} = -1.645$

$\mu = (140 + 1.645\sqrt{12\sigma^2})/12$

$= 11.76$

(b) $Y \sim N(12\mu, 12\sigma^2)$

$P(Y > 140) = 0.95 \Rightarrow \frac{140 - 12\mu}{\sqrt{12\sigma^2}} = -1.645$

$P(Y < 146) = 0.99 \Rightarrow \frac{146 - 12\mu}{\sqrt{12\sigma^2}} = 2.326$

$$\begin{cases} 140 - 12\mu = -1.645\sqrt{12\sigma^2} \\ 146 - 12\mu = 2.326\sqrt{12\sigma^2} \end{cases}$$

$3.971\sqrt{12\sigma^2} = 6 \Rightarrow \sigma^2 = 0.1902 \Rightarrow \sigma = 0.436$

$\mu = 11.874$

16. $p=0.01$

(a) if $n \geq 100$, $p \leq 0.01$ $np \leq 20$

$np = 1 < 20$.

use poisson to approximate Binormial for

$P(X=5)$ poisson $(\lambda=1)$

$= \frac{e^{-1} 1^5}{5!} = \frac{e^{-1}}{5!}$

(b) $n=200$

$\lambda = np = 2 < 20$

$P(X>3) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)$

$= 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} - \frac{e^{-2}2^2}{2!} - \frac{e^{-2}2^3}{3!}$

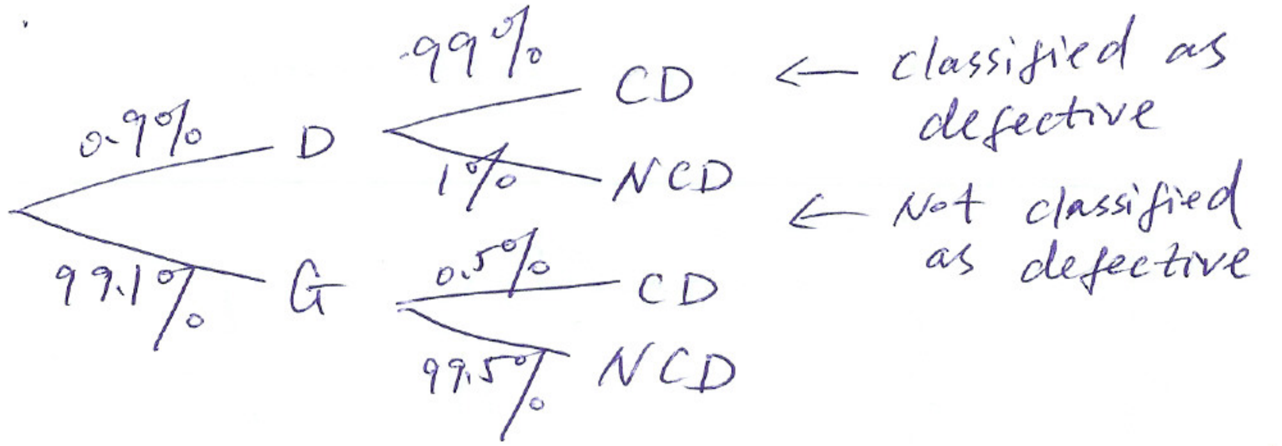
$= 1 - 0.857 = 0.143$

17. $\lambda = 2$. $X \sim \text{poisson}(2)$

(a) $P(X>5) = 1 - P(X \leq 5) = 1 - 0.983 = 0.017$

(b) $P(X=0) = \frac{e^{-2}2^0}{0!} = 0.135$

18.



(a) $P(CD) = 0.9\% \times 99\% + 99.1\% \times 0.5\%$

$= 0.01386$

(b) $P(G \mid NCD)$

$$= \frac{99.1\% \times 99.5\%}{0.9\% \times 1\% + 99.1\% \times 99.5\%}$$

$= 0.9999$