Math 2112 Solutions
Assignment 4

2.2.19 \( \exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R}^-, x > y. \)

(a) \( \forall x \in \mathbb{R} \exists y \in \mathbb{R}^- \text{ such that } x > y. \)
(b) Both the original statement and the new statement are true.

2.2.25 If the square of an integer is even, then the integer is even.

Contrapositive: If an integer is odd, then the square of the integer is odd.
Converse: If an integer is even, then the square of the integer is even.
Inverse: If the square of an integer is odd, then the integer is odd.

4.2.11 \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \text{ for all integers } n \geq 1. \)

Proof: By mathematical induction.
Base Case: Let \( n = 1. \) Then \( LHS = \frac{1}{1 \cdot 2} = \frac{1}{2} \) and \( RHS = \frac{1}{2}. \) Therefore \( LHS = RHS. \)

Inductive Step: Let \( k \geq 1. \) Assume that \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}. \)
Consider
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}
\]

By our inductive hypothesis, we know that
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}
\]
\[
= \frac{k(k+2) + 1}{(k+1)(k+2)}
\]
\[
= \frac{k^2 + 2k + 1}{(k+1)(k+2)}
\]
\[
= \frac{(k+1)(k+1)}{(k+1)(k+2)}
\]
\[
= \frac{k+1}{k+2}.
\]

But then \( P(k+1) \) is true.

Therefore \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \text{ for all integers } n \geq 1 \) by mathematical induction.
4.2.16 \( \prod_{i=0}^{n} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!} \), for all integers \( n \geq 0 \).

**Proof:** Proof by mathematical induction.

*Base Case:* Let \( n = 0 \). Then \( LHS = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2} \) and \( RHS = \frac{1}{2} = \frac{1}{2} \). Therefore \( LHS = RHS \).

*Inductive Step:* let \( k \geq 0 \). Assume that \( \prod_{i=0}^{k} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2k+2)!} \). Consider \( \prod_{i=0}^{k+1} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \). By the inductive hypothesis, we know that

\[
\prod_{i=0}^{k+1} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2k+2)!} \cdot \left( \frac{1}{2k+3} \cdot \frac{1}{2k+4} \right)
\]

\[
= \frac{1}{(2k+4)!}.
\]

But then \( P(k+1) \) is true.

Therefore \( \prod_{i=0}^{n} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!} \) for all \( n \geq 0 \).

4.3.12 For any integer \( n \geq 1 \), \( 7^n - 2^n \) is divisible by 5.

**Proof:** Proof by mathematical induction.

*Base Case:* Let \( n = 1 \). Then \( 7^1 - 2^1 = 5 \), which is indeed divisible by 5.

*Inductive Step:* Let \( k \geq 1 \). Assume that \( 7^k - 2^k \) is divisible by 5. Consider

\[
7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k
\]

\[
= 7^k + 7^k + 5 \cdot 7^k - 2^k - 2^k
\]

\[
= 7^k - 2^k + 7^k - 2^k + 5 \cdot 7^k.
\]

Note that the first two terms are divisible by 5 by our inductive hypothesis, and the last term is divisible by 5 by the definition of divisibility. Therefore \( 7^{k+1} - 2^{k+1} \) is divisible by 5.

Therefore, \( 7^n - 2^n \) is divisible by 5 for all integers \( n \geq 1 \).

4.3.30 Use mathematical induction to show that in any round-robin tournament involving \( n \) teams, where \( n \geq 2 \), it is possible to label the teams \( T_1, T_2, \ldots, T_n \) so the \( T_i \) beats \( T_{i+1} \) for all \( i = 1, 2, \ldots, n - 1 \).

**Proof:** Proof by mathematical induction.

*Base Case:* Let \( n = 2 \). Then there was only one game played. Label the winner
as $T_1$ and the loser as $T_2$.

**Inductive Step:** Let $k \geq 2$. Assume that for all tournaments with $k$ teams, there is a labelling as described above. Consider a tournament on $k + 1$ teams. When we remove one team, say team $A$, we have a tournament on $k$ teams. Thus, by our inductive hypothesis, there is a labelling $T_1, T_2, \ldots, T_k$ as above. Either $A$ beats team $T_1$, $A$ loses to the first $m$ teams (where $1 \leq m \leq k - 1$) and beats the $(m + 1)$st team, or $A$ loses to all the other teams.

In the first case, $A, T_1, T_2, \ldots, T_k$ is a desired ordering, so we relabel our teams so that $A$ becomes $T'_1$, $T_1$ becomes $T'_2$, and so on.

In the second case, $T_1, T_2, \ldots, T_m, A, T_{m+1}, \ldots, T_k$ is a desired ordering (since $A$ lost to $T_m$ but beat $T_{m+1}$), and so we relabel accordingly.

In the third case, $T_1, T_2, \ldots, T_k, A$ is a desired ordering (since $A$ lost to everyone, in particular they lost to $T_k$), and so we relabel accordingly.

In all cases, we have found the desired labelling. Thus the result holds by mathematical induction.

4.4.8 Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_2 = 3,$$

$$h_k = h_{k-1} + h_{k-2} + h_{k-3}$$

for all integers $k \geq 3$.

a. Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

b. Suppose that $s$ is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.

**a. Proof:** Proof by strong mathematical induction.

*Base Cases:* Note that $h_0 \leq 3^0, h_1 \leq 3^1, h_2 \leq 3^2$.

**Inductive Step:** Let $k > 2$. Assume that $h_i \leq 3^i$ for all integers $i$ with $0 \leq i < k$. Consider $h_k$. By our inductive hypothesis, we know that

$$h_k = h_{k-1} + h_{k-2} + h_{k-3}$$

$$\leq 3^{k-1} + 3^{k-2} + 3^{k-3}$$

$$= 3^2 \cdot 3^{k-3} + 3 \cdot 3^{k-3} + 3^{k-3}$$

$$= (3^2 + 3 + 1)3^{k-3}$$

$$\leq 3^3 \cdot 3^{k-3}$$

$$= 3^k$$
Therefore, $h_k \leq 3^k$.

Thus the result holds by strong mathematical induction.

**b. Proof:** Proof by strong mathematical induction.

Base Cases: Note that since $s > 1.83$, $h_2 \leq s^2, h_3 \leq s^3, h_4 \leq s^4$.

**Inductive Step:** Let $k > 4$. Assume that $h_i \leq s^i$ for all integers $i$ with $2 \leq i < k$. Consider $h_k$. By our inductive hypothesis, we know that

\[
h_k = h_{k-1} + h_{k-2} + h_{k-3} \\
\leq s^{k-1} + s^{k-2} + s^{k-3} \\
= s^2 \cdot s^{k-3} + s \cdot s^{k-3} + s^{k-3} \\
= (s^2 + s + 1)s^{k-3} \\
\leq s^3 \cdot s^{k-3} \\
= s^k
\]

Therefore, $h_k \leq s^k$.

Thus the result holds by strong mathematical induction.