Math 2112 Solutions
Assignment 7

7.6.23 Prove the union of two countable sets is countable.

Proof: Let A and B be countable sets. We will consider four cases.

Suppose both A and B are finite. Then $A \cup B$ is finite, and hence countable.

Suppose one of A and B is finite and the other is countably infinite. Assume without loss of generality that A is finite. Since B is countably infinite, there exists a function $f : B \mapsto \mathbb{Z}^+$ which is a one-to-one correspondence. Say that $f(b_i) = i$, for $b_i \in B$ and $i \in \mathbb{Z}^+$. Let $C = A - B$. Thus $A \cup B = B \cup C$. If $C = \emptyset$, then $A \cup B = B$ which is countably infinite. Thus assume that $C \neq \emptyset$. Suppose that $C = \{c_1, c_2, ...c_n\}$. Let $f' : B \cup C \mapsto \mathbb{Z}^+$ be defined as follows: $f'(c_j) = j$ and $f'(b_i) = n + i$. Clearly, f' is one-to-one and onto. Thus $B \cup C$ is countable. But $B \cup C = A \cup B$, so $A \cup B$ is countable.

Suppose that both A and B are infinite and that $A \cap B = \emptyset$. Given that A and B are both countable, there exists functions $f : \mathbb{Z}^+ \mapsto A$ and $g : \mathbb{Z}^+ \mapsto B$ that are one-to-one correspondences. Consider the function $h : \mathbb{Z}^+ \mapsto A \cup B$ where $h(n) = f(n/2)$ if n is even and $h(n) = g((n + 1)/2)$ if n is odd. Since both f and g are one-to-one, then so is h. Similarly, since f is onto, every element in A is covered and since g is onto, every element in B is coverd, so h is onto. Therefore, h is a one-to-one correspondence, and hence h is countable.

Suppose that both A and B are infinite and that $A \cap B \neq \emptyset$. Let $C = B - A$. Then $A \cup B = A \cup C$ and $A \cap C = \emptyset$. If C is countably infinite, then $A \cup B = A \cup C$ is countable by the previous case. If C is finite, then $A \cup B$ $A \cup C$ is countable byt the second case. In any case, $A \cup B$ is countable.

7.6.24 Use the result of 7.6.23 to prove that the set of all irrational numbers is uncountable.

Proof: Assume by way of contradiction that the set of all irrational numbers is countable. We know that the rationals are countable. Since the real numbers are the union of the irrationals and rationals, by 7.6.23, the real numbers must be countable. This contradicts a theorem discussed in class. Therefore the irrationals are uncountable.

10.2.13 Determine whether the following relation is reflexive, symmetric or transitive:
C is the circle relation on the set of real numbers, namely,

$$x, y \in \mathbb{R}, xCy \Leftrightarrow x^2 + y^2 = 1.$$

Proof: Not Reflexive: For example, $1 \not C 1$, since $1+1=2$.

Symmetric: Suppose xCy. Then $x^2+y^2=1$. But then $y^2+x^2=1$, so yCx.

Not Transitive: For example, $1C0$ and $0C1$ but $1 \not C 1$.

10.2.41 let R be a binary relation on a set A and let R^t be the transitive closure of R. Prove that for all x and y in A, xR^ty if, and only if, there is a sequence of elements of A, $x_1, x_2, ..., x_n$, such that $x = x_1, x_1Rx_2, x_2Rx_3, ..., x_{n-1}Rx_n$, and $x_n = y$.

Proof: Suppose that there is a sequence of elements of A, $x_1, x_2, ..., x_n$, such that $x = x_1, x_1Rx_2, x_2Rx_3, ..., x_{n-1}Rx_n$, and $x_n = y$. Since x_1Rx_2 and x_2Rx_3 then $x_1R^tx_3$. Since $x_1R^tx_3$ and x_3Rx_4 then $x_1R^tx_4$. Continuing onward, we can see that xR^ty.

Suppose that $x, y \in A$ and xR^ty. If xRy then we are done. Thus assume that $x \not R y$. Assume by way of contradiction that there does no exist a sequence of elements of A, $x_1, x_2, ..., x_n$ such that $x = x_1, x_1Rx_2, x_2Rx_3, ..., x_{n-1}Rx_n$, and $x_n = y$. Let $S = R^t - \{(x, y)\}$. Then S is a transitive relation that contains R and is a proper subset of R^t. This contradicts the fact that R^t is the smallest transitive relation containing R. Hense, the supposition is false, and the result follows.

10.3.16 Describe the distinct equivalence classes of:
F is the relation defined on $\mathbb{Z}$ as follows:
for all $m, n \in \mathbb{Z}$, $mFn \Leftrightarrow 4|(m-n)$.

Proof: The distinct equivalence classes are: $\{...-8,-4,0,4,8,...\}$, $\{...-7,-3,1,5,9,...\}$, $\{...-6,-2,2,6,10,...\}$ and $\{...-5,-1,3,7,11,...\}$.

10.3.21 Describe the distinct equivalence classes of:
D is the relation defined on $\mathbb{Z}$ as follows:
for all $m, n \in \mathbb{Z}$, $mDn \Leftrightarrow 3|(m^2-n^2)$.

Proof: As discussed in class, there are two distinct equivalence classes of D, namely the equivalence class consisting of all numbers divisible by three, and the equivalence class consisting of all numbers not divisible by three. This is true because if $z = 3k$, then $z^2 = 9k^2 = 3(3k^2)$, if $z = 3k + 1$, then $z^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, and finally if $z = 3k + 2$, then$z^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 6k + 1) + 1$.

10.3.33 Let R be the binary relation defined in Example 10.3.10 (the equivalence relation of rational numbers).
a. Prove that R is reflexive.
b. Prove that R is symmetric.
c. List four distinct elements in $[(1, 3)]$.
d. List four distinct elements in $[(2, 5)]$.

a. Proof: Let $(a, b) \in A$. Then $(a, b)R(a, b)$ since $ab = ba$ by commutativity of the integers.

b. Proof: Suppose that $(a, b)R(c, d)$. Therefore $ad = bc$. But then $cb = da$, so $(c, d)R(a, b)$.

c. Four distinct elements of $[(1, 3)]$ are $(1, 3)$, $(-1, -3)$, $(2, 6)$, and $(-2, -6)$ (there are, of course, many others).

d. Four distinct elements of $[(2, 5)]$ are $(2, 5)$, $(-2, -5)$, $(4, 10)$, and $(-4, -10)$.