Abel's Identity

In these notes, I will show why we can always find a solution to an initial value problem for the case of a second order linear constant coefficient homogeneous equation. Since the equation is second order, we can write it as

$$y'' + by' + cy = 0\,.$$

Let y_1 and y_2 be the two linearly independent solutions of the equation. I will define the Wronskian as

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \mathrm{y_1 y_2' - y_2 y_1'}\,.$$

I claim,

$$W = Ce^{-bt}\,,$$

for some constant C. To prove this, we note

$$\begin{aligned} W &= y_1 y_2' - y_2 y_1'\,, \\ W' &= y_1 y_2'' - y_2 y_1''\,, \\ &= y_1(-by_2' - cy_2) - y_2(-by_1' - cy_1)\,, \\ &= -b(y_1 y_2' - y_2 y_1')\,, \\ &= -bW\,. \end{aligned}$$

Thus $W = Ce^{-bt}$. The important observation to make is that either $W \equiv 0$ or $W \neq 0$ for all t.

Now if we are given an initial value problem,

$$\begin{aligned} y'' + cy' + cy &= 0\,, \\ y(t_0) &= y_0\,, \\ y'(t_0) &= y_{0p}\,. \end{aligned}$$

To find the solution $y = C_1 y_1 + C_2 y_2$, we will need to solve the linear system

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_{0p} \end{pmatrix}\,.$$

This system will have a solution provided the Wronskian is non-zero at t_0. However either the Wronskian is identically 0 or never 0. If it is identically zero, we show that y_1 and y_2 are linearly dependent (with some smoothness assumptions). If $W(y_1, y_2) \equiv 0$, then

$$\begin{aligned} y_1 y_2' &= y_2 y_1'\,, \\ \frac{y_1'}{y_1} &= \frac{y_2'}{y_2}\,, \\ \frac{d}{dx}(\ln(y_1)) &= \frac{d}{dx}(\ln(y_2))\,, \\ \ln(y_1) &= \ln(y_2) + \alpha\,, \\ y_1 &= \beta y_2\,, \end{aligned}$$

where $\alpha = \ln(\beta)$. It is clear that if two functions are linearly dependent, then the Wronskian must be identically 0. So two functions are linearly dependent if and only if the Wronskian is nonzero somewhere (with some smoothness conditions). If we have two linearly independent solutions, we can then find a unique solution to any initial value problem

We can easily extend this proof to general second order linear equations

$$y'' + p(t)y' + q(t)y = 0\,.$$

In this case, the same argument will show that

$$W(y_1, y_2) = Ce^{-\int p(t)\,dt}\,.$$

And again we can conclude that either $W \equiv 0$ or $W \neq 0$ for any t.