Epidemic Models - Hysteresis

1 SIR models

In these models, we divide a population into compartments. We will consider the following subsets of a given population of N individuals

- S - the susceptible population.
- I - the infected population.
- R - the recovered population (assumed to have immunity).

We will use a law of mass action type of rate to model the transfer of infection to the susceptible group of the population

$$\dot{S} = \underbrace{bN}_{\text{birth}} - \underbrace{bS}_{\text{death}} - \underbrace{\beta\frac{SI}{N}}_{\text{infection}}, \tag{1}$$

$$\dot{I} = \underbrace{\beta\frac{SI}{N}}_{\text{infection}} - \underbrace{bI}_{\text{death}} - \underbrace{\nu I}_{\text{recovery}}, \tag{2}$$

$$\dot{R} = \underbrace{\nu I}_{\text{recovery}} - \underbrace{bR}_{\text{death}}. \tag{3}$$

Here we have assumed the birth rate and the death rate are the same b. The total population is then constant. This is clear if we let $N = S + I + R$ then $\dot{N} = 0$. We may interpret the parameters in the model as follows:

- b - birth and death rate.
- N - total population size.
- β - per ca pita infection rate.
- nu - recovery rate.

To determine if there is an epidemic, we look at the stability of the disease free equilibrium (DFE) $(S, I, R) = (N, 0, 0)$. The Jacobian of the system is given by

$$J = \begin{pmatrix} -b - \beta\frac{I}{N} & -\beta\frac{S}{N} & 0 \\ \beta\frac{I}{N} & \beta\frac{S}{N} - b - \nu & 0 \\ 0 & \nu & -b \end{pmatrix}.$$

If we sub in the DFE we have,

$$J = \begin{pmatrix} -b & -\beta & 0 \\ 0 & \beta - b - \nu & 0 \\ 0 & \nu & -b \end{pmatrix}.$$

We can find the eigenvalues of this matrix quite easily. They are given by

$$\lambda_1 = -b\,, \tag{4}$$
$$\lambda_2 = \beta - b - \nu\,, \tag{5}$$
$$\lambda_3 = -b \tag{6}$$

Since λ_1 and λ_3 are always negative, the stability of the DFE is determined by λ_2. If we define $R_0 = \frac{\beta}{b+\nu}$, then we have

$$\lambda_2 = (b+\nu)(R_0 - 1)\,.$$

So if $R_0 < 1$, the DFE is stable and if $R_0 > 1$ the DFE is unstable. The constant R_0 is call the basic reproduction number and it represents the expected number of infected individuals resulting from a single infected case.

2 Model with hysteresis

We will now consider a slightly more complex model in which the contact rate depends on the number of infected individuals. As well we will allow for the loss of immunity in the recovered class. The model is then,

$$\dot{S} = \underbrace{bN}_{\text{birth}} - \underbrace{bS}_{\text{death}} \underbrace{-\beta(1+\mu I)\frac{SI}{N}}_{\text{infection}} + \underbrace{\gamma R}_{\text{loss of immunity}}\,, \tag{7}$$
$$\dot{I} = \underbrace{\beta(1+\mu I)\frac{SI}{N}}_{\text{infection}} - \underbrace{bI}_{\text{death}} - \underbrace{\nu I}_{\text{recovery}}\,, \tag{8}$$
$$\dot{R} = \underbrace{\nu I}_{\text{recovery}} - \underbrace{bR}_{\text{death}} - \underbrace{\gamma R}_{\text{loss of immunity}}\,. \tag{9}$$

The per capita infection rate here is given by $\beta(1+\mu I)$ and is a function of the number of infected. Details of this model may be found in the paper ”Epidemic Solutions and Endemic Catastrophies” P. van den Driessche and James Watmough Proceedings of an International Workshop on Dynamical Systems and their Applications in Biology Fields Institute Communications, August 2-6, 2001, pp. 247-258.

First we define the constants

$$R_0 = \frac{\beta}{b+\nu}\,, \quad \epsilon = \frac{\nu}{b+\gamma}\,, \quad \eta = \frac{\nu}{b+\nu}\,, \quad \chi = \frac{\mu N}{1+\epsilon}\,.$$

We rescale the system

$$x = \frac{1+\epsilon}{N} I\,, \quad y = (1+\epsilon)\frac{R}{N}\,, \quad t' = (\nu + b)t\,,$$

and use $S = N - I - R$ to get the reduced system. I will go through the details to find the equation for x. We will let x' be the derivative with respect to t'. So we have,

$$
\begin{aligned}
x' &= \frac{dx}{dt}\frac{dt}{dt'}\,, \\
&= \frac{1}{\nu+b}\frac{dx}{dt}\,, \\
&= \frac{1}{\nu+b}\frac{1+\epsilon}{N}\frac{dI}{dt}\,, \\
&= \frac{1}{\nu+b}\frac{1+\epsilon}{N}\left(\beta\left(1+\mu\left(\frac{N}{1+\epsilon}x\right)\right)\left(N-\frac{N}{1+\epsilon}(x+y)\right)\frac{1}{N}-\frac{N}{1+\epsilon}(b+\nu)x\right)\,,
\end{aligned}
$$

So after some cancelling we have,

$$
\begin{aligned}
x' &= R_0(1+\chi x)\left(1-\frac{x+y}{1+\epsilon}\right)x - x\,, \\
y' &= \eta\left(x-\frac{y}{\epsilon}\right)\,.
\end{aligned}
$$

For this system we have a DFE and R_0 is the basic reproduction number. We now examine the endemic equilibrium. For the endemic equilibrium we require $y = \epsilon x$. Subbing this into the first equation, we find that

$$
R_0(1+\chi x)(1-x) = 1\,,
$$

or

$$
R_0 f(x) = 1\,,
$$

where

$$
f(x) = (1+\chi x)(1-x)
$$

. Solving the quadratic gives

$$
x^{\pm} = \frac{\chi-1}{2\chi} \pm \frac{1}{2\chi}\sqrt{(\chi+1)^2 - \frac{4\chi}{R_0}}\,.
$$

So if $(\chi+1)^2 > \frac{4\chi}{R_0}$, then we have 2 addition endemic equilibria. To determine the stability of these equilibria we sub them into the Jacobian

$$
J = \begin{pmatrix} R_0 x^{\pm}\left(\chi(1-x^{\pm}) - \frac{1+\chi x^{\pm}}{1+\epsilon}\right) & -R_0 x^{\pm}\frac{1+\chi x^{\pm}}{1+\epsilon} \\ \eta & -\frac{\eta}{\epsilon} \end{pmatrix}\,.
$$

We note that

$$
Det(J) = -R_0\eta x^{\pm}\frac{f'(x^{\pm})}{\epsilon}\,,
$$

where f is defined above. The determinant will be less than zero on x^- and positive on x^+. So as long as the trace is negative, the upper branch of equilibria will be stable and the lower will be unstable. To demonstrate what is occurring we set $\chi = 2$. The two equilibria are then

$$x^{\pm} = \frac{1}{4}\left(1 \pm \sqrt{9 - \frac{8}{R_0}}\right).$$

So as long as $R_0 > \frac{8}{9}$, we will have three distinct equilibria. Note that at $R_0 = 1$, $x^- = 0$. So for $R_0 > 1$, we can ignore x^- as it will be less than zero and unphysical.

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In the above graph, we plot the various equilibria and their stability for a range of R_0 values. The red lines represent stable equilibria and the blue unstable. There are several key points to consider.

1. For $\frac{8}{9} < R_0 < 1$, we have a stable endemic equilibrium as well as a stable DFE. So even though $R_0 < 1$, we can have an outbreak if there is enough infected individuals at time $t = 0$. Basically if $I(0) < \frac{N}{1+\epsilon}x^-$ the disease will die out in this case. However if $I(0) > \frac{N}{1+\epsilon}x^-$, then the infected class will grow to x^+. Note that we must scale back to the original variables.

2. In most epidemic models, as R_0 passes through 1, a small endemic infected class forms. In this case, as R_0 passes through 1, the infected class quickly jumps to $\frac{N}{1+\epsilon}x^+$.

3. Once we are on the x^+ branch of equilibria, bringing R_0 back below 1 will not be enough to return us to the DFE. In this case we must reduce R_0 below $\frac{8}{9}$.

This phenomenon is referred to as hysteresis or catastrophe theory.