Stability of Numerical Methods

We will consider two types of stability for numerical methods:

**zero-stability** Growth of round-off errors.

**a-stability** Growth of rapidly decaying modes of the solution

To study zero-stability, we consider the test problem

\[ y' = 0, \quad y(0) = a. \]

The exact solution is a constant and we would like the numerical approximation to result in a constant solution. If we consider a 1-step method applied to this test problem, we find

\[ y_{n+1} = y_n. \]

If we assume a solution of the form \( y_i = \zeta^i \), we get

\[ \zeta^n (\zeta - 1) = 0. \]

We have one root of \( \zeta = 1 \), which we must have for the method to be consistent, the other roots are all 0. As long as all the other root satisfy \( \zeta_i \leq 1 \), the method is zero stable.

We will now apply this method to a multi-step method. Consider the third order Backwards Difference Formula:

\[ y_{n+3} - \frac{18}{11} y_{n+2} + \frac{9}{11} y_{n+1} - \frac{2}{11} y_n = \frac{6}{11} h f(t_{n+3}, y_{n+3}). \]

If we apply our this method to our test problem, we get

\[ y_{n+3} - \frac{18}{11} y_{n+2} + \frac{9}{11} y_{n+1} - \frac{2}{11} y_n = 0. \]

Now we let \( y_i = \zeta^i \), then we have

\[ \zeta^3 - \frac{18}{11} \zeta^2 + \frac{9}{11} \zeta - \frac{2}{11} = 0. \]

The three roots are

\[ \zeta_1 = 1, \quad \zeta_2 = \frac{7 + \sqrt{39}i}{22}, \quad \zeta_3 = \frac{7 - \sqrt{39}i}{22}. \]

Here \( |\zeta_{1,2}| = \frac{\sqrt{7}}{11} \sim 0.426 < 1 \), so this method is stable.

Here is another linear multi-step scheme considered in the book.

\[ y_{n+1} + \frac{3}{2} y_n - 3 y_{n-1} + \frac{1}{2} y_{n-2} = 3 h f(t_n, y_n) \]

This is a 3rd order consistent method. If we check for zero-stability, we will get the polynomial

\[ \zeta^3 + \frac{3}{2} \zeta^2 - 3 \zeta + \frac{1}{2} = 0. \]

The roots are given by

\[ \zeta_1 = 0, \quad \zeta_2 = \frac{-5 + \sqrt{33}}{4}, \quad \zeta_3 = \frac{-5 - \sqrt{33}}{4}. \]

In this case, \( |\zeta_3| \sim 2.686 > 1 \), so this method isn’t zero-stable. This method can’t be used for any problems.

**Absolute Stability**

The other form of stability that we will consider is absolute stability. We consider the test problem,

\[ y' = \lambda y. \]

We would like the approximate solutions to decay when \( \text{Re}(\lambda) < 0 \) We define the region of the complex \( h \lambda \) plane where the approximate solution satisfies \( |y_{i+1}| < |y_i| \). Here \( h \) is the step-size.
For example we will find the region for the mid-point method,

\[ y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)). \]

We apply this method to our test problem,

\[
\begin{align*}
y_{i+1} &= y_i + h\lambda \left( y_i + \frac{h}{2}\lambda y_i \right), \\
y_{i+1} &= y_i \left( 1 + h\lambda + \frac{(h\lambda)^2}{2} \right).
\end{align*}
\]

The region of absolute stability is then defined by

\[ \left| 1 + z + \frac{z^2}{2} \right| \leq 1. \]

To find the region, we plot the solutions to \( 1 + z + \frac{z^2}{2} = e^{i\theta} \) as \( \theta \) ranges from 0 to \( 2\pi \). We can plot this in Matlab using the roots command. Look up the help page for it. The session is given below.

```matlab
octave:8> t=linspace(0,2*pi);
octave:9> for j=1:100
    > p=[1/2 1 1-exp(i*t(j))];
    > plot(roots(p),'x');
    > hold on
    > end
octave:10> print("rk2stab.eps")
octave:11> diary
```

Note that we must be careful with the variable \( i \). If we were to use it as an index, we could no longer use it as an imaginary number. The resulting graph is in Figure 1. Just to try something new, let's find the region for the standard 4th order method with the table,

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<th>1/2</th>
<th>1/2</th>
<th>1/2</th>
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<td>1</td>
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<td>1/2</td>
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<td>1/2</td>
</tr>
</tbody>
</table>

If we apply this method to our test problem carefully, we get

\[ y_{n+1} = y_n \left( 1 + h\lambda + (h\lambda)^2 + \frac{(h\lambda)^3}{2} + \frac{(h\lambda)^4}{4} \right). \]

So we want to find the roots of the polynomial

\[ 1 + z + z^2 + \frac{1}{2}z^3 + \frac{1}{4}z^4 = e^{i\theta} \]

as \( \theta \) ranges from 0 to \( 2\pi \). Here is the Matlab session:
octave:2> t=linspace(0,2*pi);
octave:3> for j=1:100
> p=[1/4 1/2 1 1 1-exp(i*t(j))];
> plot(roots(p),'x');
> hold on
> end
octave:4> print("rk4stab.eps")
octave:5> diary

![Region of Absolute Stability](rk4stab.eps)

Figure 2: The region of absolute stability for the standard 4th order Runge-Kutta method

The process for determining the region of absolute stability for a multi-step scheme is a little different. For example we consider Addams-Bashford 2.

\[ y_{i+1} = y_i + \frac{3h}{2} f_i - \frac{h}{2} f_{i-1} \]

We apply this method to our test problem,

\[ y_{i+1} = y_i + \frac{3}{2} h\lambda y_i - \frac{h}{2} h\lambda y_{i-1} \]

We seek solutions to this relation with \( y_i = \zeta^i \) with \( \zeta = e^{\theta} \).

\[ \zeta^{i+1} = \zeta^i + \frac{3}{2} h\lambda \zeta^i - \frac{1}{2} h\lambda \zeta^{i-1} \]
\[ \zeta^2 = \zeta + \frac{3}{2} \zeta h\lambda - \frac{1}{2} h\lambda \]
\[ h\lambda = \frac{\zeta^2 - \zeta}{\frac{3}{2} \zeta - \frac{1}{2}} \]

Now we just plot \( h\lambda \) as \( \theta \) goes from 0 to 2\( \pi \).

octave:2> t=linspace(0,2*pi);
octave:3> z=(exp(2*i.*t)-exp(i.*t))./(1.5*exp(i.*t)-.5);
octave:4> plot(z,'x')
octave:5> print("abstab.eps")
octave:6> diary

As I have discussed implicit methods have much better stability characteristics then explicit methods. This often makes them the better choice even though implicit methods require us to solve a non-linear system. Consider the 3rd order BDF method,

\[ y_{n+3} = \frac{18}{11} y_{n+2} + \frac{9}{11} y_{n+1} - \frac{2}{11} y_n = \frac{6}{11} hf_{n+3} \]

We apply this method to our problem and solve for \( h\lambda \). I will set \( y_n = \zeta^n \) as well.

\[ \zeta^3 - \frac{18}{11} \zeta^2 + \frac{9}{11} \zeta - \frac{2}{11} = \frac{6}{11} h\lambda \zeta^3 \]
\[ h\lambda = \frac{\zeta^3 - \frac{18}{11} \zeta^2 + \frac{9}{11} \zeta - \frac{2}{11}}{\zeta^3} \]
We can now use Matlab to plot the stability region.

```matlab
octave:2> t=linspace(0,2*pi);
octave:3> z=(exp(3.*i.*t)-18/11.*exp(2.*i.*t)+9/11.*exp(i.*t)-2/11)./exp(3.*i.*t);
octave:4> plot(z,'x')
octave:5> print("bdf3stab.eps")
octave:6> diary
```

Here is the resulting figure. The region of absolute stability is outside of the shape.

Figure 4: The region of absolute stability for the BDF 3 method is the outside of the enclosed shape. This method is almost A-stable. The two small lobes cross slightly into the left half plane.