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TRAFFIC DYNAMICS: STUDIES IN CAR FOLLOWING

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The manner in which vehicles follow each other on a highway (without passing) and the propagation disturbances down a line of vehicles has been investigated. Experimental data is presented which indicates that the acceleration at time t of a car which is attempting to follow a leader is proportional to the difference in velocity of the two cars at a time $(t-\Delta)$, Δ being about 1.5 sec and the proportionality constant being about 0.37 sec^{-1}. It is shown theoretically that the motion of a long line of vehicles becomes unstable when the product of the lag time and the proportionality constant exceeds one-half. The experimental data implies that driving is done on the verge of instability. A variety of other laws of following is analyzed theoretically.

THE VITAL DEPENDENCE of our daily activities on the efficient and safe flow of vehicular traffic has stimulated the accumulation of enormous amounts of relevant empirical data by traffic engineers.[1] These data and the parallel research in road construction have been the basis of the development of our modern highways. However, it is only recently that serious thought has been devoted to the analysis of the fundamental mechanisms which operate to control the movement of traffic.

Several interesting theoretical approaches to the characterization of these mechanisms have been proposed. A review of these, as well as an extensive bibliography, has been given by GERLOUGH AND MATHEWSON.[2]

PIPES[3] has studied the dynamics of a linear array of vehicles whose

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motion is characterized by rules given in the California Motor Vehicle Code Summary, namely, "a good rule for following another vehicle at a safe distance is to allow yourself the length of a car (about 15 feet) for every ten miles an hour you are traveling." He showed how lines of cars stop and start and perform other following operations on the assumption that responses are immediate and that no inertial effects exist in the vehicles or response lags in the operators. He also discusses several other mechanisms of following. Similar analyses have also been made by REUSCHEL.[4]

LIGHTHILL AND WHITHAM[5] and RICHARDS[6] have postulated the density of traffic on a long highway to be a continuous function of position along the highway and of time. The traffic is then treated as a fluid flowing along the highway. The mathematical methods of fluid dynamics have been applied to a discussion of various highway phenomena, such as the development of shock waves when sudden stops and starts are made. PRAGER[7] has made a two-dimensional continuum model of the flow of traffic in large areas, such as cities.

NEWELL[8] has stressed the analogy between the motion of vehicles on a sparsely populated highway and the behavior of molecules in rarified gases. The motion of both is a 'free flow' except during occasional encounters with other elements. When a fast car overtakes a slow one, the encounter usually results in a loss of time, namely, that required for the passing operation, or an equivalent reduction in the mean velocity of the fast car. Occasionally the opposite effect occurs when a driver on a low density highway speeds up in preparation for and during the passing operation.

Considerable interest exists in the simulation of traffic with high speed computers. For example, GERLOUGH AND MATHEWSON[2] and GOODE[9] have been simulating the behavior of vehicles at road intersections.

Although the fluid flow approach mentioned above shows considerable promise of providing a framework for a general theory of traffic, we feel that it is worthwhile to investigate the possible application of another highly developed branch of modern applied mathematics, namely, the theory of servomechanisms and network analysis. In its most general form this theory is merely that of the analysis of the propagation of assorted signals through 'black boxes' arranged in various topological configurations. In traffic analysis we might consider individual vehicles or certain sets of vehicles as the signals and the highway as the network.

An important 'black box' in a traffic network is an intersection with or without a traffic light. The four outputs, the traffic leaving the intersection in four directions at time t, are related to the four inputs, vehicles approaching the intersection during some time interval $t-\tau$. The dependence of the outputs on the inputs characterizes the intersection 'black box.'

The manner in which a given length of intersectionless highway fits into the black box pattern can be seen by considering a two-lane highway. Suppose two types of vehicles are using the highway—low speed trucks and high speed passenger cars. First consider the case of traffic flowing in opposite directions in the two lanes with the occasional passing of low by high speed vehicles. At low traffic densities only a small amount of time is lost in passing so that the output of fast vehicles in one lane is simply related to the input of both fast and slow vehicles of the same lane at some previous time interval. As the traffic density increases an interaction develops between the flow in the two lanes—opportunities for passing become rarer and the output of a given lane is related to the input of both lanes (and perhaps also to the output of the other lane since a jam in the second lane prevents passing in the first). Finally, as the density becomes very high no passing can occur. In the case of both lanes of traffic proceeding in the same direction, the output of fast vehicles from a given length of highway depends on the input of all types. The resistance to flow of fast vehicles depends increasingly on the number of slow ones as the over-all density increases, since a passenger car trapped behind a truck in the slow lane has difficulty in escaping when other passenger cars are whizzing by in the fast lane. The detailed relations between inputs and outputs in a stretch of highway gives the characteristics of a schematic black box that might be used in a network analysis.

Once the characteristics of the elements of the traffic network are understood, we can expect to be able to employ some analogies between traffic and communication theory, since one of the main problems of a communications engineer is to pass as much information on a given circuit per unit time as possible while the traffic engineer attempts to pass as many vehicles as possible. As in communication theory, various sources of noise exist in traffic theory, e.g., pedestrians.

Instabilities of two types exist in traffic—traffic jams and accidents. Of the two kinds of accidents the spontaneous (caused by such driver failure as falling asleep or committing errors in judgment, and such mechanical failure as blowouts) and the inherent (which results from the accumulation of small effects over which nobody has complete control and leads to systems instability), only the second is amenable to some theoretical analysis (the first being statistical in nature).

A driver programs his driving operations in various ways. In the absence of other interfering vehicles, he attempts to keep his speed fairly constant at a set point determined by a compromise between the urge to minimize trip duration and maximize safety. When following other vehicles whose speed is of the order of his set point speed the driver introduces a new set point, the inter-car spacing whose value depends on his speed.

The servomechanism approach is especially useful in clarifying the role and interaction of the three components of the traffic system—the road topology (number of lanes, nature of intersection, signals, warning signs, etc.), the vehicle characteristics (speed, acceleration and deceleration qualities, signaling mechanisms, vision, etc.) and the operator's behavior (range of perception, lags between perception and response, etc.). This approach gives one the opportunity of making the study of traffic an experimental as well as an observational science.

One can set up artificial traffic situations to correspond to various elements or 'black boxes' in the traffic network and by controlling the nature of the inputs the dependence of outputs on inputs might be established with greater dispatch than is possible by a detailed analysis of traffic on real highways. We are optimistic enough to believe that the dynamics of real traffic can be synthesized from results of experiment and theory. One of the results of this type of investigation is that quantitative information might be obtained on the effect of the introduction of new signaling devices on cars and roads and of the behavior of abnormal drivers (tired, drunk, etc.) on the elements of the traffic network. Finally if the vehicle of the future is to be automatic as well as automobile, its design can only follow an understanding of the traffic system as a servomechanism.

This paper is our first discussion of a traffic element treated as a servomechanism. We consider the theory of the manner in which one car follows another, and are especially interested in determining the conditions required for stable following. We shall propose various models of the game of 'Follow that car!' and compare such models with experimental data on how cars are actually followed. There is some merit in studying models which do not correspond to general practice since some of these may be more stable (and safer) and might be put into use by installing appropriate signaling devices on cars. The theory discussed here is not limited to automobile traffic but might be applied to other 'follow the leader situations.' We hope in future publications to discuss a variety of traffic network elements and to make remarks about complete traffic systems.

THEORY OF FOLLOW THE LEADER

Accidents caused by improper following can occur in two ways. If a driver follows the car in front so closely that he cannot avoid an accident caused by a sudden perturbation, he has merely been using bad judgment and no mathematical analysis is required. However, accidents frequently occur in collisions which involve cars considerably behind the car that initiated some fluctuation. It is such accidents that may result in the multiple car pile-ups which are sometimes observed on congested superhighways, especially at high speeds. It is this latter case that results from an amplification of the original perturbation as it is transmitted down the line of traffic.

Let us consider a line of identical vehicles that are attempting to follow each other in a steady or stable manner. We assume that if such a state could be achieved, the separation distance between vehicles plus the car length would have a constant value* a and each vehicle would have the same velocity v. The spacing a would in general depend on v. We let $u_n(t)$, the deviation of the velocity of the nth vehicle from the velocity v, be given by

$$u_n(t) = dx_n/dt - v, \tag{1}$$

where x measures distance and $y_n(t)$ the spacing of vehicles given by

$$y_n(t) = x_{n-1}(t) - x_n(t). \tag{2}$$

As the operator of the nth vehicle observes variations in $u_n(t)$ or $y_n(t)$, he applies either his accelerator or brakes to keep from lagging or closing in on his leader. Two factors prevent this operator from immediately reproducing the leader motions. His delayed response and that of the mechanisms which transmit brake and acceleration signals to the vehicle contribute a lag in the follow-the-leader process as does the inertia of the vehicle itself.

The accelerating force (other than the force required to maintain the steady motion) applied to the nth vehicle at time t can be expected to depend on its instantaneous velocity deviation $u_n(t)$ and on some functional of the difference in velocities of the $(n-1)$st and nth vehicles $u_{n-1}(\tau) - u_n(\tau)$ (for some range of τ with $\tau \leqq t$) as well as on a functional of the spacing $y_n(\tau)$. The equations of motion of the individual vehicles assumed to have the same mass, M, are then given by

$$M\, du_n(t)/dt = F\{u_n(t);\ f_1[u_{n-1}(\tau) - u_n(\tau)];\ f_2[y_n(\tau)]\}, \tag{3}$$

where $u_0(t)$ refers to the velocity pattern of the lead vehicle.

In the past, vehicle operating data has not been analyzed with a view to determining the precise form of the functional F. We shall discuss the results of preliminary experiments carried out for this purpose later in this paper. The purpose of the present section is to investigate the stability characteristics of various choices of the functional F. Even though some of these forms may not be generally prevalent in automobile operation today, some knowledge of their consequences may be of interest in that they indicate dangerous types of behavior and might suggest new

* When the mean separation distance is very large each driver tends to behave independently and the theory developed is no longer applicable. We are concerned primarily with the high traffic density situation in which no passing is allowed. We hope to develop a phenomenological theory of passing at a later time.

forms of signaling devices for the improvement of responses. The development of the automatic automobile of the future will require an understanding of the follow-the-leader process. The mathematical models given below are linear. As will be pointed out later linear equations appear to give surprisingly good agreement with an experiment that corresponds to the high density follow-the-leader case. The introduction of a nonlinear functional causes no fundamental difficulty in solving the equations of motion. This is so because the equation of motion for a particular vehicle depends only on the behavior of its predecessor so that the equations can be solved successively. Complications would arise if the influence of vehicles other than nearest neighbors were included.

Proportionate Control

As a first example we postulate that the applied force is proportional to the instantaneous difference in the velocity of a given vehicle and its predecessor, or the case of 'proportional control' in the language of servomechanism theory. The equations of motion of a line of N identical vehicles each of mass M is

$$M\,du_n/dt=\lambda\,(u_{n-1}-u_n), \qquad (n=1, 2, \cdots, N) \quad (4)$$

where λ is the sensitivity of the control mechanism. At instants in which a lead car is going faster than the following car, the follower applies an accelerating force and vice versa. We assume in equation (4) and throughout this paper that the sensitivities for acceleration and deceleration are identical. Although this is a reasonable approximation in a properly functioning car at low speed, it is certainly not the case at high speed or when for example either the brakes are poor or an engine is not well tuned. The solution of these equations depends on the velocity pattern, $u_0(t)$, of the lead vehicle. The stability of a line of traffic depends on whether a local fluctuation in velocity is damped out or amplified as it propagates down the line of cars. There are two types of instability, local and asymptotic instability. We are concerned with the latter. It should be noted that even asymptotic stability conditions depend on the equilibrium spacing and velocity. If the equilibrium spacing is small, then one does not have to go back far in the line of vehicles behind the initial perturbation to find the occurrence of a collision. Although from our solutions of the equations of motion one can determine where down the line an accident occurs we are primarily interested in the criteria for the growth or decay of a disturbance.

Since the system now under consideration is linear, this stability question can be investigated in terms of the Fourier components of the driving function $u_0(t)$. Let us assume that the driving function is monochromatic with the frequency ω so that

$$u_0(t)=e^{i\omega t}. \tag{5}$$

Of course an arbitrary driving function can be expressed as a linear combination of monochromatic components by the usual Fourier analysis. By substituting

$$u_n(t)=f_n\, e^{i\omega t}, \qquad f_0=1, \tag{6}$$

into equation (4) we find

$$(i\omega M/\lambda)\, f_n=f_{n-1}-f_n, \tag{7}$$

so that

$$f_n=(1+i\omega M/\lambda)^{-n}\, f_0, \tag{8}$$

and

$$u_n(t)=(1+\omega^2M^2/\lambda^2)^{-n/2}\exp\{i\,[\omega t-n\cos^{-1}(1+\omega^2M^2/\lambda^2)^{-1/2}]\}. \tag{9}$$

The amplitude of the velocity deviation decreases with increasing n for all frequencies, masses, and sensitivities. Hence instantaneous proportional control is stable under all circumstances. The phase velocity of a signal of frequency ω, in terms of car spacings per second is

$$\frac{dn}{dt}=\omega\cos^{-1}(1+\omega^2M^2/\lambda^2)^{-1/2}\cong\begin{cases}\omega^2M/\lambda & \text{for}\quad \omega\rightarrow 0\\ \omega\pi/2 & \text{for}\quad \omega\rightarrow\infty.\end{cases} \tag{10}$$

The spacing between the $(n+1)$st and nth vehicle is

$$\begin{aligned} y_n(t)&=y_n(t_0)+\int_{t_0}^{t}[u_{n-1}(t)-u_n(t)]\,dt,\\ &=y_n(t_0)+(M/\lambda)(1+i\omega M/\lambda)^{-n}\,(e^{i\omega t}-e^{i\omega t_0}).\end{aligned} \tag{11}$$

Even though the decay of y_n with n implies asymptotic stability the amplitude of say y_1 might be sufficiently large to cause local instability. Suppose one chooses t_0 to be a time at which the spacing $y_n(t_0)$ has the normal value a. Then the greater the sensitivity λ, the more stable the spacing for all t and n. In principle one would like to make λ as large as possible. However, we shall see below that time lags in control systems limit the sensitivity λ for stable driving. Qualitatively the limitation results from the fact that if both the lag and λ are large, then large corrective measures are taken for observed variations whose effects might die out more quickly than the time required for the responses to make themselves felt.

Response Lag

Equation (4) can be generalized to include the lag in the response of the operator through the introduction of a weight function $\sigma(t)$. Then the following relation

$$M\,\frac{du_n}{dt}=\int_0^\infty [u_{n-1}(t-\tau)-u_n(t-\tau)]\,d\sigma(\tau), \tag{12}$$

indicates that the total force applied at a given time t depends on a weighted average of all earlier differences in u_{n-1} and u_n. The choice

$$\sigma(\tau)=\lambda\,H(\tau-\Delta)=\begin{cases}0, & (\tau<\Delta)\\ \lambda, & (\tau>\Delta)\end{cases} \tag{13}$$

where H is the Heaviside step function, or

$$\sigma'(\tau)=\lambda\,\delta(\tau-\Delta), \tag{13a}$$

$\delta(x)$ being the Dirac delta function, corresponds to a time lag Δ between the observation of a velocity difference and the application of a correcting force. Equation (12) then becomes

$$M\,du_n(t)/dt=\lambda\,[u_{n-1}(t-\Delta)-u_n(t-\Delta)]. \tag{14}$$

As before we let $u_0(t)=e^{i\omega t}$,

and substitute equation (6) into equation (14). We then find

$$(i\omega M/\lambda)\,e^{i\Delta\omega}f_n=f_{n-1}-f_n, \tag{15}$$

or

$$2\,i\mu\omega\,e^{i\Delta\omega}f_n=f_{n-1}-f_n, \tag{15a}$$

where $\mu=M/(2\,\lambda)$,

so that

$$f_n=(1+2\,i\mu\omega\,e^{i\Delta\omega})^{-n}f_0, \tag{16}$$

and

$$u_n(t)=(1+4\,\mu^2\omega^2-4\,\mu\omega\,\sin\Delta\omega)^{-n/2}\times\exp\{i\,[\omega t-n\,\cos^{-1}(1+4\,\mu^2\omega^2-4\,\mu\omega\,\sin\Delta\omega)^{-1/2}]\}. \tag{17}$$

The amplitude factor decreases with increasing n if

$$1+4\,\mu^2\omega^2-4\,\mu\omega\,\sin\Delta\omega>1,$$

i.e., if

$$4\,\mu^2\omega>4\,\mu\,\sin\omega\Delta. \tag{18}$$

Low frequencies give the greatest limitations on sensitivities. As $\omega\to 0$, λ must satisfy the inequality

$$\lambda<M/(2\,\Delta), \tag{19}$$

or

$$\Delta<\mu. \tag{19a}$$

Hence, for a given lag Δ, a stable operation results as long as the inequality is satisfied.

As in the previous case the spacing between the $(n-1)$st and nth vehicle is given by

$$y_n(t)=y_n(t_0)+2\,\mu[e^{i\omega(t+\Delta)}-e^{i\omega(t_0+\Delta)}]/(1+2\,i\mu\omega\,e^{i\Delta\omega})^n. \qquad (20)$$

A more realistic response function $\sigma(t)$ is one with a dead period lag followed by a continuous response

$$\sigma(\tau)=\begin{cases}\lambda(1-e^{-(\tau-\Delta)/\delta}), & (\tau>\Delta)\\ 0 & (\tau<\Delta)\end{cases} \qquad (21)$$

Then our fundamental equation becomes

$$M\,\frac{du_n(t)}{dt}=\frac{\lambda}{\delta}\int_{\Delta}^{\infty}[u_{n-1}(t-\tau)-u_n(t-\tau)]\,e^{-(\tau-\Delta)/\delta}\,d\tau. \qquad (22)$$

Differentiation of equation (22) with respect to t yields

$$M\,\frac{d^2u_n(t)}{dt^2}=-\frac{\lambda}{\delta}\int_{\Delta}^{\infty}e^{-(\tau-\Delta)/\delta}\,\frac{d}{d\tau}\,[u_{n-1}(t-\tau)-u_n(t-\tau)]\,d\tau,$$

so that after integration by parts we find

$$M\,\frac{d^2u_n(t)}{dt^2}+\frac{M}{\delta}\,\frac{du_n(t)}{dt}=\frac{\lambda}{\delta}\,[u_{n-1}(t-\Delta)-u_n(t-\Delta)]. \qquad (23)$$

We again set $u_n(t)=f_n\,e^{i\omega t}$ and find

$$u_n(t)=\left\{\frac{\lambda\,e^{-i\omega\Delta}}{\lambda\,e^{-i\omega\Delta}+i\omega M-\delta M\omega^2}\right\}^n e^{i\omega t}. \qquad (24)$$

It is easy to show by the methods discussed above that fluctuations in our line of traffic will be damped out rather than amplified if

$$M\omega\,(1+\delta^2\omega^2)>2\,\lambda\,[\sin\omega\Delta+\omega\delta\,\cos\omega\delta].$$

As before, the most restrictive condition on time lags and relaxation times exists at low frequencies. Stability exists at all frequencies if

$$\lambda<M/[2(\delta+\Delta)]. \qquad (25)$$

Notice that the time lag Δ and the relaxation time δ are additive in determining stability conditions.

Constant Spacing

A mode of driving that is unstable even without control-response lags is that in which an operator attempts to keep the distance between vehicles constant and applies a force proportional to the deviation of this distance from the required spacing when fluctuations occur. We introduce a moving coordinate system which progresses with the mean velocity of the lead car and has its origin at the position the lead car would have if it always moved with this velocity. Then, if a is the required spacing, we let $x_n(t)$ be the deviation of the position of the nth vehicle from the point $-an$ in the moving coordinate system.

The equations of motion of a line of vehicles that employs this mode of control are

$$M\, d^2x_n/dt^2 = K\,(x_{n-1} - x_n). \tag{26}$$

Again suppose $x_0(t) = e^{i\omega t}$

and $x_n(t) = f_n e^{i\omega t}$;

then $-M\omega^2 f_n = K\,(f_{n-1} - f_n)$,

so that

$$f_n = (1 - MK^{-1}\omega^2)^{-n} f_0 \tag{27}$$

or

$$x_n(t) = (1 - MK^{-1}\omega^2)^{-n}\, e^{i\omega t}. \tag{27a}$$

Note that for any value of ω a resonance condition exists when

$$\omega = (K/M)^{1/2}, \tag{27b}$$

so that fluctuations in separation distance would be amplified. This situation is of importance when a group of cars follows one another at very small velocity independent distances such as occurs frequently on our super highways during rush hours. Then a fluctuation in position of one car amplifies down the line of cars and can cause an accident if the line of cars is sufficiently long.

California Code

A control scheme whose stability is rather insensitive to lags can be devised by following a rule suggested in the California Vehicle Code Summary:[3] "A good rule for following another vehicle at a safe distance is to allow yourself the length of a car (about fifteen feet) for every ten miles per hour you are traveling." This rule implies that

$$x_{n-1} = x_n + b + T\, v_n + L_{n-1},$$

where b is the standard distance between vehicles at rest, L_n is the length of the nth vehicle, and T is the time constant inferred by the California Code [$T \cong 15$ ft/(14.67 ft/sec)$\cong 1$ sec]. We assume L_n to be a constant, $c - b$, for all vehicles and write

$$x_{n-1} = x_n + c + T\, v_n. \tag{28}$$

Fluctuations in lead car performance would, as a result of various response lags, cause equation (28) to be violated in spite of the best intentions of followers. If

$$\delta_n(t) = x_{n-1}(t) - x_n(t) - c - T\, v_n(t) > 0, \tag{29}$$

the nth driver would accelerate in order to recover the equality in equation (28) and vice versa. Let us suppose that at any time t a force proportional to $\delta_n(t-\Delta)$ is applied to the nth car. Then the equations of motion of our line of vehicles are

$$M\, d^2x_n/dt^2 = K\,[x_{n-1}(t-\Delta) - x_n(t-\Delta) - c - T\, dx_n(t-\Delta)/dt]. \quad (30)$$

The constant c can be eliminated by letting

$$x_n = x_n' - ctT^{-1}.$$

Then x_n' satisfies the equation

$$M\, d^2x_n'(t)/dt^2 = K\,[x'_{n-1}(t-\Delta) - x_n'(t-\Delta) - T\, dx_n'(t-\Delta)/dt]. \quad (31)$$

As usual, we investigate stability by letting $x_n'(t) = f_n\, e^{i\omega t}$. Then

$$f_n = (1 + i\omega T - MK^{-1}\omega^2\, e^{i\omega\Delta})^{-n},$$

and our stability criterion is

$$T^2 + (MK^{-1}\omega)^2 > 2\, MK^{-1}\,[\cos\omega\Delta + \omega T \sin\omega\Delta]. \quad (32)$$

The low frequency condition, $\omega \rightarrow 0$,

$$T^2 > 2\, M/K \quad (33)$$

is sufficient to insure stability at all frequencies independently of the lag, Δ. Note that if this condition is not satisfied, resonances might occur. Suppose the lag Δ is very small then equation (32) becomes

$$T^2 + (MK^{-1}\omega)^2 > 2\, MK^{-1}\,[1 - \tfrac{1}{2}\,\omega^2\Delta^2 + \omega^2 T\Delta]$$

or

$$\omega^2\,[M^2K^{-2} + MK^{-1}\Delta^2 - T\Delta] > 2\, MK^{-1} - T^2. \quad (34)$$

Hence, if equation (33) is not satisfied, resonances occur at frequencies w for which the absolute value of the denominator in equation (31a) vanishes:

$$1 + \omega^2 T^2 + M^2K^{-2}\omega^4 = 2\, MK^{-1}\omega^2\,(\cos\omega\Delta + T\omega \sin\omega\Delta), \quad (35)$$

which reduces to equation (27b) when $T = \Delta = 0$.

The inequality $T^2 > 2\, MK^{-1}$ is to be interpreted as meaning that stability exists for any sensitivity K provided that the time constant T is made sufficiently large. Remember that a large value of T implies a conservative or greater spacing between cars.

Propagation of a Perturbation

One can follow the details of the propagation of a perturbation down a line of cars through the use of the Laplace transform. As an example let us suppose that the dynamical equations are those given in equation (12) and that no disturbance in velocity occurs for $t<0$. Then

$$M\frac{du_n}{dt}=\int_0^t[u_{n-1}(t-\tau)-u_n(t-\tau)]\,d\sigma(\tau),\qquad (n=1,2,\cdots)\quad (12)$$

Furthermore we assume that the velocity variation of the first car from the average velocity $\bar{v}$ is given by

$$u_0(t)=f(t),\qquad (36)$$

where $f(t)=0$ if $t<0$. We define the Laplace transforms of $u_n(t)$, $\sigma(t)$, and $f(t)$, respectively, to be

$$U_n(s)=\int_0^\infty u_n(t)\,e^{-st}\,dt,\qquad (37)$$

$S_1(s)$ and $F(s)$. Then it can be shown that

$$U_n(s)=\left[\frac{S_1(s)}{sM+S_1(s)}\right]^n f(s).\qquad (38)$$

The standard Laplace transform inversion formula yields

$$u_n(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\left[\frac{S_1(s)}{sM+S_1(s)}\right]^n e^{st}\,ds,\qquad (39)$$

while the spacing between the $(n-1)$st and nth car is given by

$$y_n(t)=a+\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\left[\frac{S_1(s)}{sM+S_1(s)}\right]^n[S_1(s)]^{-1}\,(e^{-st}-1)\,ds,\qquad (40)$$

where a is the normal spacing.

As an example of the application of equation (39) we consider the propagation of disturbances in a system with a dead period lag Δ. Then using equation (13a) and

$$S_1(s)=\lambda\,e^{s\Delta},$$

we have

$$u_n(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(s)\left[\frac{1}{2\,\mu s\,e^{-s\Delta}+1}\right]^n e^{st}\,ds.\qquad (41)$$

Let us assume that no singularities exist in the integrand in the right half plane Re $s\geqq 0$. Then we can set $c=0$ and $s=i\omega$ to obtain

$$u_n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(i\omega)\,[1+2\,i\mu\omega e^{-i\omega\Delta}]^{-n}\,e^{i\omega t}\,d\omega.\qquad (42)$$

One can show that if the stability condition $\lambda<M/2\Delta$ is satisfied the quantity in square brackets in equation (42) achieves its maximum ab-

solute value when $\omega=0$. Hence when n is large we can expect values of ω near 0 to give the main contribution to $u_n(t)$. In this region

$$[1+2\,i\mu\omega\,e^{-i\omega\Delta}]^{-1}=\exp\{-2\,i\mu\omega-2\,\mu\,(\mu-\Delta)\,\omega^2+O(\omega^3)\}, \quad (43)$$

so that

$$u_n(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[f(0)+i\omega\,f'(0)+\cdots]\exp[i\omega(t-2\,\mu n)] \times\exp[-2\,\mu\,(\mu-\Delta)\,n\omega^2+O(n\omega^3)]\,d\omega. \quad (44)$$

If we let

$$z=[2\,\mu n\,(\mu-\Delta)]^{1/2}\,\omega, \quad (45)$$

then, as $n\to\infty$,

$$u_n(t)\cong f(0)\,[8\,\pi^2\mu n\,(\mu-\Delta)]^{-1/2}\int_{-\infty}^{\infty}e^{-z^2}\exp\left\{i\,\frac{t-2\,\mu n}{[2\,\mu n\,(\mu-\Delta)]^{1/2}}\right\}y\,dy, \quad (46)$$

and finally

$$u_n(t)\cong f(0)\,[8\,\pi\mu n\,(\mu-\Delta)]^{-1/2}\exp\left\{-\frac{(t-2\,\mu n)^2}{8\,\mu n\,(\mu-\Delta)}\right\}. \quad (47)$$

This shows that under stable conditions the low frequency component of a disturbance is transmitted over the greatest distance. The velocity of propagation, in number of car separations per unit time, is

$$n/t=\lambda/M=1/(2\,\mu). \quad (48)$$

As a result of experiments that will be described later the quantity λ/M is of the order of 0.4 $\sec^{-1}$ for a typical modern vehicle used in the experiment. The width of a time pulse is of the order of

$$[2\,\mu n\,(\mu-\Delta)].^{1/2} \quad (49)$$

Notice that as the lag Δ increases (i.e., as $\Delta\to\mu$) the amplitude of $u_n(t)$ grows until instability is reached when $\Delta=\mu$. When $\Delta>\mu$ in the unstable range, equation (47) is no longer valid because the denominator of the integrand of equation (46) has a pole to the right of the imaginary axis. When one wishes to follow the details of the development of an instability in a line of cars separate integrations of equation (41) must be made for each value of n.

Velocity-Dependent Sensitivity

It should be pointed out that it would be surprising if the sensitivity λ were velocity independent. Suppose as a rough correction we assume that

$$\lambda=\lambda_0\,(1+\alpha v). \quad (50)$$

Then in a range of small velocity variations about the average $\bar{v}$ the stability condition in equation (19) becomes

$$\lambda_0 (1+\alpha\bar{v}) < M/(2\Delta). \qquad (51)$$

Hence if the velocity coefficient α were positive λ_0 would have to be reduced with increasing velocity to preserve stability.

Emergency Control

When two cars become closer than some critical distance, X (whose value might depend on the velocity of the second car), an emergency decelerating force is applied by the second car to prevent a collision. The operator would slam on his breaks to give the maximum deceleration mechanically feasible. The law of following might then be approximated by

$$\ddot{x}_n(t) = \alpha\,[\dot{x}_{n-1}(t-\Delta) - \dot{x}_n(t-\Delta)][1-H(z_n)] - \beta\,H(z_n)$$

where H is the Heaviside step function defined in equation (13) and

$$z_n \equiv X - x_{n-1}(t-\Delta') + x_n(t-\Delta').$$

The differential equations become nonlinear and although they can be solved analytically, their solution is clumsy. We have therefore programmed them for machine solution. The new parameters β, X, and Δ' must be determined experimentally.

EXPERIMENTS AND THEIR INTERPRETATION

In order to obtain statistical estimates of certain functions and parameters for a preliminary evaluation of the mathematical models previously mentioned, it was necessary to design and conduct an experimental study to collect quantitative information regarding driver-car performance in a two-lane highway in which one car cannot pass another owing to the high traffic density in the opposite direction.

We now give a brief discussion of the experimental apparatus employed in the experiment and consider the process of one car following another without passing. Let $x_l(t)$ and $x_f(t)$ be the positions of the lead and following car at a time t so that the spacing between the cars is $x_l - x_f$. Also let the velocities of the respective cars be represented by v_l and v_f so that the relative velocity of the two cars is $v_l - v_f$.

To measure the spacing and the relative velocity of the two cars, a car follower, which is shown in Fig. 1, was designed and installed in a test car. The car follower consists essentially of a reel and a power unit mounted on a small platform which was fastened on the front bumper of the test car. Several hundred feet of fine wire were wound on the reel, and the end of the wire was fastened on the rear bumper of a lead car. A constant wire tension was maintained by means of a slipping friction clutch.

Inasmuch as the power unit kept the wire very taut at all times, $x_l - x_f$ was measured by the position of the reel at any particular instant, which depends on the amount of wire stretched between the two cars. This measurement was made by using a multiple turn potentiometer geared to

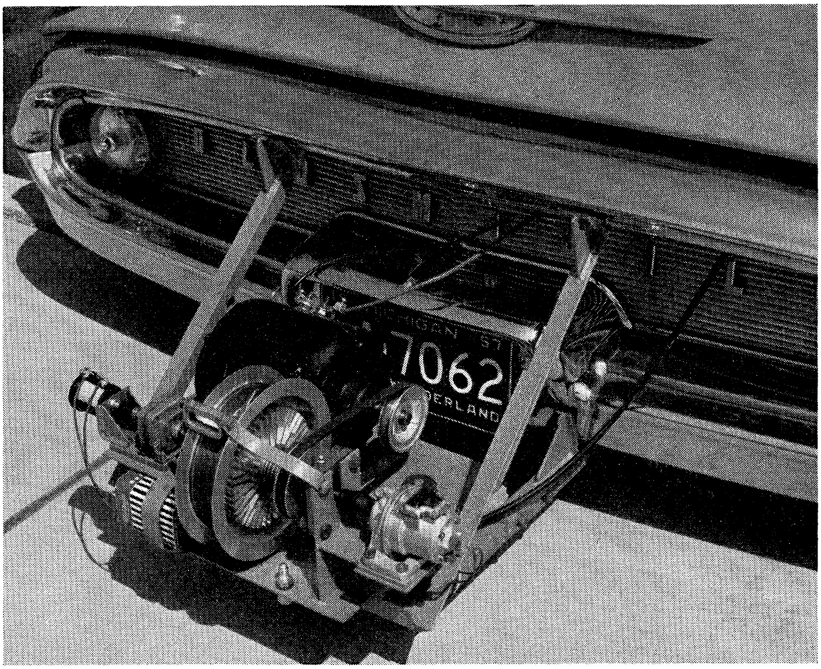

Fig. 1. Photograph of car follower showing wire reel and power unit.

a reel shaft. A direct current generator tachometer operating off the same shaft gave a measure of the rate at which the wire was wound or unwound, which is proportional to $v_l - v_f$. A fifth wheel attached to the test car measured v_f, while an accelerometer mounted in the car indicated the car's longitudinal acceleration which is designated by a_f.

The totality of this information, i.e., $x_l - x_f$, $v_l - v_f$, v_f, and a_f, was recorded simultaneously by an oscillograph installed in the back seat of the test car.

Eight male drivers participated in the study. These people, all employees of the Research Staff of the General Motors Technical Center, ranged in age from 24 to 38 years. Prior to testing each subject drove the test car, a 1957 Oldsmobile, until he indicated that he was sufficiently familiar with the car's response, controls, etc., to operate the car safely in congested traffic. Each driver then operated the car behind a lead car in an actual experimental run on the test track at the General Motors Technical Center. Testing time was approximately 20 to 30 minutes per driver.

The directions given to the drivers were simply, "Follow the lead car at what you consider to be a minimum safe distance at all times."

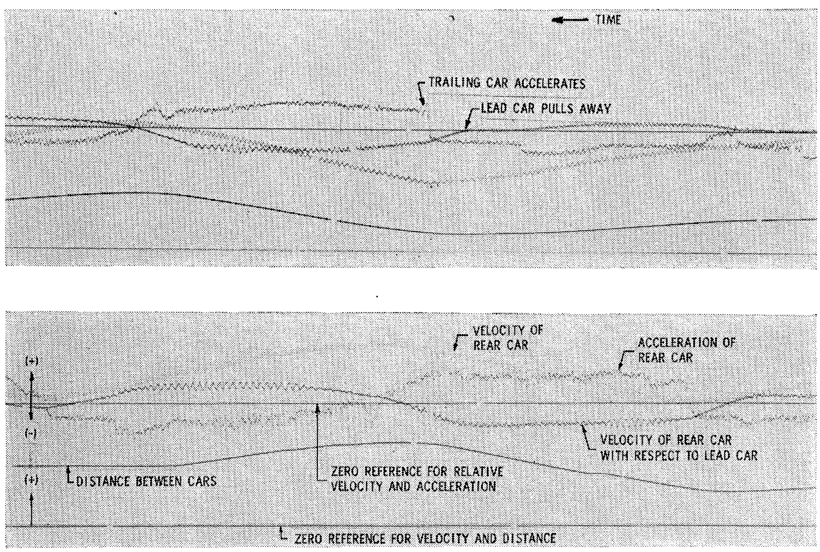

Fig. 2. The oscillograph recording shown below identifies the various curves recorded in the car-following experiments. The top strip is a typical recording from such an experiment.

These directions were employed in an attempt to produce a driving situation that would evoke driver behavior similar to that which might be observed as a person drives in dense traffic. The driver of the lead car, in all cases, pursued no prescribed program or driving pattern, but randomly varied his speed within the range of 10 to 80 mph and included several braking actions.

The information recorded on the oscillograph was of the type shown in Fig. 2. The records were inspected to identify a continuous section in each record where the test conditions were more or less dynamic. In other words, sections of the records in which spacing, $x_l - x_f$, and speed, v_f, are constant, are trivial and of no interest in the present study.

The aim of our data analysis was to obtain a relation between the acceleration, a_f, the relative velocity, $v_l - v_f$, and the spacing, $x_l - x_f$, of the form

$$a_f = f_1(v_l - v_f) + f_2(x_l - x_f). \tag{52}$$

(Note that we recorded $v_f - v_l$ on the tracings shown in Fig. 2 for ease of measurement.) The analysis was made by reading points equally spaced in time from the relevant parts of the three curves on the experimental records. The functions f_1 and f_2 were first assumed to be linear and by the method of least squares a multiple correlation coefficient was derived from the record of each driver.

It was discovered that the space dependent function $f_2(x_l - x_f)$ did not contribute significantly to the correlation. Consequently, this function was dropped from equation (52). Since the choice of a linear form of $f_1(v_l - v_f)$ with the omission of f_2 led to relatively high correlation coefficients in the neighborhood of 0.80–0.90, and in view of the preliminary character of our experiment, it was deemed unnecessary to examine nonlinear forms for the f's.

An appreciation of the physical factors involved in the experimentation dictates that the best linear correlation would be achieved through the introduction of a time lag Δ. Hence our statistical problem was to determine the values of the constants b and Δ, which yield the best least squares fit to the equation

$$a_f(t) = b\,[v_l(t-\Delta) - v_f(t-\Delta)]. \tag{53}$$

Correlation procedures for this type of analysis have been recently reviewed by MERRILL AND BENNETT.[10] The relation in equation (53) is exactly that given in equation (14) and the constant b is identified as λ/M.

The lag constant Δ is the sum of three more elementary lags. We note that the $(v_l - v_f)$ can be regarded as stimuli input to the driver, i.e., the information which tells him to effect a change in his car's acceleration. After an acceleration change is made by the driver of the lead car, the response of the trailing vehicle depends upon its driver's perception time, t_1, his response time, t_2, and the time of the response of the vehicle, t_3. Inasmuch as each driver-car combination has its own parameters, Δ and λ/M, we readily discern the necessity for limiting our present discussion to the particular eight drivers and the test car used in this experiment.

Since we do not know the individual t_i's, we can let Δ take on various values. Then by plotting Δ versus the correlation coefficient, r, we can identify an optimum for each driver. The constants b and Δ for a given driver are those associated with the maximum of his r versus Δ curve and are given in Table I.

The fact that the mean value $(2\lambda\Delta/M)_{\mathrm{AV}} = 1.12$ is so close to unity shows that the model of follow-the-leader given by equation (14) is a fairly accurate description of the dynamics of a line of cars. Although the stability condition $\Delta/\mu<1$ is violated slightly, the degree of violation is within the experimental error. Two extra stabilizing influences exist in actual highway traffic. A given driver generally notices the behavior of the vehicle two ahead of him as well as that which follows him (through a rear view mirror or horn signals by his follower).

It would be interesting to extend the experiments here described to more extended lines of cars to evaluate the degree of coupling of a car with rear and second nearest front neighbors and to introduce these interactions into the dynamical equations. Of course new stability conditions would

TABLE I

PARAMETERS OF EQUATION (53)

Driver	Δ ($r=\max$)	$b=\lambda/M$	r	$2\lambda\Delta/M$
1	1.4 sec	0.74 sec^{-1}	0.87	2.08
2	1.0	0.44	0.90	0.88
3	1.5	0.34	0.86	1.03
4	1.5	0.32	0.49	0.97
5	1.7	0.38	0.74	1.29
6	1.1	0.17	0.86	0.37
7	2.2	0.32	0.82	1.43
8	2.0	0.23	0.85	0.93
Average......	1.55	0.368		1.12

result. Anyone who has done considerable driving notices that the margin between stable and unstable operation is very narrow. In practice one would expect that even the added stabilizing influences would yield values of the appropriate parameters in the dynamical equations, which would make driving conditions merely a shade on the stable side.

A few conservative drivers interspersed in a chain of cars add tremendously to the stability because they effectively cut the chain by leaving such large gaps that disturbances that might have grown earlier in the chain have time to damp out. In dense traffic such gaps are however soon filled by their more impatient brethren so that their good influence is frequently nullified.

It is to be emphasized that a phenomenological theory of traffic dynamics lumps together a large number of mechanical and human attributes that can only with great difficulty be handled individually. This, however, is what makes the use of phenomenological models so powerful in the unravelling of so complicated a set of events.

TOPICS FOR FUTURE INVESTIGATION IN THE THEORY OF TRAFFIC FLOW ON THE UNLIMITED HIGHWAY

THE PREVIOUS sections of this paper have been concerned with dense traffic situations in which no passing is possible. We close with a few remarks on passing, bunching, and acceleration noise, topics that we hope to discuss both theoretically and experimentally in future publications.

In the high density limit, passing can be treated as a queuing problem. Suppose a fixed obstacle exists on a two-lane highway. Cars that accumulate behind the obstacle are only able to go around it when appearance time gaps larger than a certain critical value exist between successive vehicles in the opposite lane. If large gaps (which allow two or more cars to go around the obstacle per gap) are rare the rate of growth of the line behind the obstacle and the reduction in traffic flow current caused by the obstacle can be discussed by standard queuing theory. The distribution of service times of the queue is the distribution of time intervals between the required long gaps. The distribution of appearance times is of course that of the time intervals between the appearance of successive cars at the obstacle. A slow driver is a moving obstacle and can be treated in the same manner as a fixed one through the use of a moving coordinate system. When very large gaps are common so that two or more cars may occasionally pass the obstacle together the queuing theory becomes more complicated. Queuing theory is also applicable to the analysis of the effect of a bad curve or very steep grade on traffic flow.

An alternative approach to the passing problem can be made by setting up continuum flow equations for each lane, including cross terms which characterize the interactions between the lanes.

Another effect caused by the existence of a speed distribution in medium density traffic conditions is bunching. Everyone has seen clusters of cars form and evaporate. It would be interesting to observe the distribution of cluster sizes as a function of mean speed and density and to find the gel point at which clusters congeal to form a jammed traffic situation.

The estimation of the state of the traffic on an open road is a highly personal matter. The driver who is satisfied in maintaining a speed of 35 mph while his fellow travelers are racing along at 70 mph considers them to be lunatics. The speedier drivers consider our snail to be a menace. A resistance to flow caused by speed dispersion can be defined in a different subjective way for each driver. The mean resistance averaged over all drivers might serve as a useful parameter of the traffic stream. A quantity sensitive to the resistance to flow is the acceleration noise experienced by a given vehicle. We define this noise as the dispersion in the acceleration distribution function. The only measurement we have of this quantity at the moment is that obtained from the records of the follow-the-leader experiments discussed earlier. The acceleration distributions are essentially Gaussian with mean zero and dispersion of the order of $\sim 0.15\ g$. We plan to make more extensive measurements of this quantity under real and well-specified highway conditions.

Finally, a car moving with the average speed of the stream would have a very narrow acceleration distribution pattern, while one that moves faster than the average stream speed would be expected to have a broadened acceleration distribution that would increase with the speed differential. One might try to relate the resistance of the stream to the acceleration noise of its component cars. A car moving with a speed lower than the stream average will cause a reduction of the stream velocity, the magnitude of the reduction increasing with the density.

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