Math 4190/5190 -Differential Equations-Qualitative Theory
Homework #2 Due Monday Oct 8

1. Show that if two vector fields are smoothly equivalent, then the eigenvalues of the linearization at corresponding equilibria are the same. (Bonus question for 4190)

2. Consider the system of FitzHugh-Nagumo PDE's, which model nerve signal propagation along an axon:

$$u_t = u_{xx} - u(u-a)(u-1) - v\,,$$
$$v_t = bu\,,$$

where $-\infty < x < \infty$, $t > 0$ are the independent variables and $0 < a < 1$, $b > 0$ are parameters. Solutions of the form

$$u(x,t) = U(\zeta)\,, \quad v(x,t) = V(\zeta)\,, \quad \zeta = x + ct\,,$$

where c is an unknown wave propagation speed, are called *travelling waves*.

(a) Derive a system of three first-order ODE's (called the *wave system*) for U, V and $W = \dot{U}$ with "time" ζ.

(b) for $c > 0$, show that the wave system has a unique equilibrium with one positive eigenvalue λ_1, and two eigenvalues λ_2, λ_3 with negative real parts. (Hint: first verify this assuming the eigenvalues are real. Then show that the characteristic equation cannot have roots on the imaginary axis. Finally use the continuous dependence of the eigenvalues on the parameters.)

(c) Conclude that the equilibrium has a one-dimensional local unstable manifold and a two-dimensional local stable manifold. Assume that there are parameter values for which a *homoclinic* orbit exists (a non-constant orbit $(U(\zeta), V(\zeta), W(\zeta))$ which approaches the equilibrium as $\zeta \to \pm\infty$). Sketch possible profiles of the corresponding $U(\zeta)$ (consider the cases where λ_2, λ_3 are real and distinct, and where they are complex conjugates), and explain why these correspond to travelling *impulse* solutions to the PDE's.

3. Let (r, θ) denote polar coordinates in the plane.

(a) Consider the parameterized family of vector fields

$$\dot{r} = \alpha r - r^3\,,$$
$$\dot{\theta} = \omega + r^2 \sin(4\theta)\,,$$

where α is a parameter, $-\infty < \alpha < \infty$ and ω is a fixed positive constant. Find all equilibria and discuss their linearized stabilities. Draw the global phase portraits and indicate the stable and unstable sets of the equilibria and periodic orbits (there are several cases, depending on the value of α). For what values of α is there *i*) a periodic orbit? *ii*) and invariant circle?

(b) Sketch the phase portraits for the parameterized family of mappings

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} (1+\alpha)r - r^3 \\ \theta + \omega \mod 2\pi \end{pmatrix}$$

for $|r| < \epsilon$, where α is a parameter, $-\epsilon^2 < \alpha < \epsilon^2$, and ω is a fixed positive constant. Here $\epsilon > 0$ is chosen and fixed sufficiently small so that the mappings are local diffeomorphisms.

Also, for $0 < \alpha < \epsilon^2$, show that if ω is a rational multiple of 2π, there are infinitely many cycles, and otherwise there are no (non-constant) cycles.

4. Fix δ, k, and $\omega > 0$ and consider the parametrically excited damped pendulum

$$\ddot{\theta} + \delta\dot{\theta} + (k^2 + \epsilon\cos(\omega t))\sin\theta = 0\,, \quad \theta \in S^1\,, \quad \dot{\theta} \in \mathbb{R}\,.$$

Using two-dimensional Poincaré maps and perturbing from $\epsilon = 0$, discuss the existence and stability of periodic solutions for small nonzero ϵ.