Math 4190/5190 -Differential Equations
Homework #4 Due Monday November 26

1. Consider the system
$$\dot{x} = Ax + f(x)\,, \quad x \in \mathbb{R}^2\,,$$
where
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,,$$
f is C^3 and $f(x) = O(|x|^2)$. Find a near identity transformation $x = h(y) = y + h_2(y)$, where the components of h_2 are homogeneous polynomials of degree two in $y = (y_1, y_2)$, that transforms the system into Poincare normal form up to second order
$$\begin{aligned} \dot{y_1} &= y_2 + O(|y_1, y_2|^3)\,, \\ \dot{y_2} &= ay_1^2 + by_1y_2 + O(|y_1, y_2|^3)\,. \end{aligned}$$
Find the Poincare normal form coefficients a and b in terms of partial derivatives of f at $x = 0$. Sketch phase portraits of the *truncated* transformed system for various a and b assuming they are both nonzero.

2. In this problem we consider a model (the Brusellator) for a chemical reaction. If the container is well stirred, diffusion of chemicals can be ignored, and the equation takes the form,
$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2y\,, \\ \dot{y} &= bx - x^2y\,, \end{aligned}$$
where a, b, x and y represent concentrations if chemicals. a and b are assumed to remain constant, so are considered to be parameters. Here we fix $a > 0$ and treat b as a (positive) bifurcation parameter.
   (a) Find the unique equilibrium, linearize about the equilibrium and show that the linearization has pure imaginary eigenvalues when $b = 1 + a^2$.
   (b) Analyze the Hopf bufurcation that occurs at $b = 1 + a^2$. Draw a branching diagram, and sketch local phase portraits (up to topological equivalence) fo the Brusellator for $b < 1 + a^2$, $b = 1 + a^2$ and $b > 1 + a^2$, b near $1 + a^2$.

3. Consider the one-parameter family of vector fields:
$$\begin{aligned} \dot{x_1} &= 2x_1 + (2+\alpha)x_2\,, \\ \dot{x_2} &= x_1 + x_2 + x_1^4\,. \end{aligned}$$
where α is a parameter. Use centre manifold theory to study the dynamics near the origin, for α near 0 ($< 0, = 0, > 0$). Determine the dynamics on the centre manifold, using the implicit function theorem to justify the neglect of higher order terms. Draw two-dimensional phase portraits for α near 0, and draw a branching diagram.