Math 4200/5200 -Differential Equations-Qualitative Theory
Homework #1 Due Wed Jan 25

1. (a) Sketch phase portraits for $\dot{x} = \alpha - x^2$, $x \in \mathbb{R}$, where α is a constant parameter, in the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$.

   (b) Draw curves in the αx-plane showing the locations of equilibria $x^0 = x^0(\alpha)$. Use a solid curve to denote a "branch" of stable equilibria and a broken curve to denote a branch of unstable equilibria.

   (c) Find explicit expressions for the local flows $x(t; x_0)$ and the maximal intervals of existence $\mathcal{I}_{x_0}$ in each of the three cases. Sketch graphs of representative solutions and check that they are consistent with the phase portraits drawn in (a).

2. (a) Find e^{tA} ($t \in \mathbb{R}$) and A^k ($k \in \mathbb{Z}$), if A is the 3×3 elementary Jordan block,

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

   (b) Find e^{tA} ($t \in \mathbb{R}$), if A is the 4×4 elementary real block,

$$\begin{pmatrix} \mu & -\omega & 1 & 0 \\ \omega & \mu & 0 & 1 \\ 0 & 0 & \mu & -\omega \\ 0 & 0 & \omega & \mu \end{pmatrix}.$$

   (c) Find explicit 2×2 real matrices A, with all eigenvalues μ satisfying $|\mu| = 1$, such that,
      i. 0 is a stable fixed point for $x_{n+1} = Ax_n$;
      ii. 0 is an unstable fixed point for $x_{n+1} = Ax_n$;

3. Classify all real 2×2 matrices A according to whether the origin is a *i*) hyperbolic sink, *ii*) hyperbolic saddle, *iii*) hyperbolic source or *iv*) non-hyperbolic equilibrium for $\dot{x} = Ax$ (work in terms of the eigenvalues). Plot a diagram in the $\sigma\Delta$-plane showing the classification, where σ is the trace (sum of the elements along the main diagonal) and Δ is the determinant. In addition, sketch phase portraits for all cases (up to topological equivalence) where the origin is a non-hyperbolic equilibrium.

4. Classify all real 2×2 matrices A, $\det A \neq 0$, according to whether the origin is a *i*) hyperbolic sink, *ii*) hyperbolic saddle, *iii*) hyperbolic source or *iv*) non-hyperbolic fixed point for $x \mapsto Ax$ (work in terms of eigenvalues). Plot a diagram in the $\sigma\Delta$-plane showing the classification, where σ is the trace and Δ is the determinant.