Math 4200/5200 -Differential Equations-Qualitative Theory
Homework #2 Due Friday Feb 10

1. Consider
$$\dot{x} = f(x,\alpha,\epsilon) = f_0(x,\alpha) + \epsilon f_1(x,\alpha,\epsilon)\,, \quad x,\alpha,\epsilon \in \mathbb{R}$$
where $f_0(0,0) = 0$, $f_{0,x}(0,0) = 0$, $f_{0,\alpha}(0,0) \neq 0$, $f_{0,xx}(0,0) \neq 0$ and $f_1(x,\alpha,\epsilon)$ is an arbitrary (sufficiently smooth) function. For $\epsilon \neq 0$, this can be viewed as a perturbed fold bifurcation that occurs when $\epsilon = 0$.
   (a) Use the implicit function theorem to prove that for all sufficiently small ϵ, there is a non-hyperbolic equilibrium $x = x^0(\epsilon)$ when $\alpha = \alpha^0(\epsilon)$, where x^0 and α^0 are smooth functions with $x^0(0) = 0$ and $\alpha^0(0) = 0$. Thus the bifurcation conditions for a fold bifurcation occur along a curve $\alpha^0(\epsilon)$ in the $\alpha\epsilon$ plane
   (b) Show that $f_\alpha(x^0(\epsilon),\alpha^0(\epsilon),\epsilon)$ and $f_{xx}(x^0(\epsilon),\alpha^0(\epsilon),\epsilon)$ have the same signs as $f_{0,\alpha}(0,0)$ and $f_{0,xx}(0,0)$ respectively for sufficiently small ϵ. Thus for fixed $\epsilon \neq 0$, the perturbed family undergoes a fold bifurcation as α increase past $\alpha^0(\epsilon)$, of the same type as the unperturbed problem as α increases past 0. Draw a bifurcation diagram in the $\alpha\epsilon$ plane indicating the phase portraits, in the case $f_{0,\alpha}(0,0) > 0$, $f_{0,xx}(0,0) > 0$.

2. The Transcritical bifurcation. Consider
$$\dot{x} = f(x,\alpha)\,, \quad x \in \mathbb{R}\,,\, \alpha \in \mathbb{R}$$
where $f(0,0) = 0$ and $f_x(0,0) = 0$. In addition we assume the constraint
$$f(0,\alpha) = 0$$
for all α.
   (a) If f satisfies nondegeneracy conditions
$$f_{x\alpha}(0,0) \neq 0 \quad f_{xx}(0,0) \neq 0\,,$$
   show that $\dot{x} = f(x,\alpha)$ if locally topologically equivalent to
$$\dot{y} = \beta y + \sigma y^2\,,$$
   where $\sigma = \pm 1$. Draw all the possible branching diagrams of $\dot{x} = f(x,\alpha)$, depending on the signs of $f_{x\alpha}(0,0)$ and $f_{xx}(0,0)$. The bifurcation which occurs is called transcritical.
   (b) Show that by choosing an appropriate constant $y_0(\beta)$, the change of variables $y = y_0(\beta) + u$ takes $\dot{y} = \beta y + \sigma y^2$ into a family of the form $\dot{u} = \gamma + \sigma u^2$. What is the relation between the parameters β and γ? How can the transcritical bifurcation be described in terms of a one-parameter path in the bifurcation diagram of the fold bifurcation?
   (c) Following the ideas of question 1, show that transcritical bifurcations persist under all sufficiently small perturbations satisfying the constraint that $x = 0$ is an equilibrium for all parameter values.
   (d) Show that transcritical bifurcations do not persist under all perturbations, by giving an explicit example of an arbitrary small perturbation that destroys a particular transcritical bifurcation. Draw branching diagrams to illustrate the structural instability of the transcritical bifurcation.

3. Consider the logistic map, a simple population dynhamics model:

$$x_{n+1} = f(x_n, \alpha) = \alpha x_n(1 - x_n)\,,\; x_n \in \mathbb{R}\,,\; \alpha \in \mathbb{R}\,.$$

(a) Show that there is a flip bifurcation when $\alpha = 3$. As α increases past three, a stable fixed point becoms unstable, while a stable period two cycle bifurcates from this point.

(b) Show that there is a fold bifurcation of period 3 cycles when $\alpha = 1 + \sqrt{8}$. As α increases past $1 + \sqrt{8}$, a stable and unstable pair of period 3 cycles is generated.

Note: For students enrolled in 4200, questions 2 b), c) and d) are bonus questions.