

Math 4200/5200 -Differential Equations-Qualitative Theory

Homework #3 Due Friday March 5

1. Consider the system

$$\dot{x} = Ax + f(x), \quad x \in \mathbb{R}^2,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

f is C^3 and $f(x) = O(|x|^2)$. Find a near identity transformation $x = h(y) = y + h_2(y)$, where the components of h_2 are homogeneous polynomials of degree two in $y = (y_1, y_2)$, that transforms the system into the normal form up to second order

$$\begin{aligned} \dot{y}_1 &= y_2 + O(|y_1, y_2|^3), \\ \dot{y}_2 &= ay_1^2 + by_1y_2 + O(|y_1, y_2|^3). \end{aligned}$$

Find the normal form coefficients a and b in terms of partial derivatives of f at $x = 0$. Sketch phase portraits of the *truncated* transformed system for various a and b assuming they are both nonzero.

2. In this problem we consider a model (the Brusellator) for a chemical reaction. If the container is well stirred, diffusion of chemicals can be ignored, and the equation takes the form,

$$\begin{aligned} \dot{x} &= a - (b + 1)x + x^2y, \\ \dot{y} &= bx - x^2y, \end{aligned}$$

where a , b , x and y represent concentrations of chemicals. a and b are assumed to remain constant, so are considered to be parameters. Here we fix $a > 0$ and treat b as a (positive) bifurcation parameter.

- (a) Find the unique equilibrium, linearize about the equilibrium and show that the linearization has pure imaginary eigenvalues when $b = 1 + a^2$.
- (b) Analyze the Hopf bifurcation that occurs at $b = 1 + a^2$. Draw a branching diagram, and sketch local phase portraits (up to topological equivalence) for the Brusellator for $b < 1 + a^2$, $b = 1 + a^2$ and $b > 1 + a^2$, b near $1 + a^2$.

Note: For students enrolled in Math 4200 the first question will count as a bonus.