1. Suppose that f is analytic in a domain containing $\{z : |z| \leq 1\}$ and that $|f(z)| < 1$ for all z on the unit circle $|z| = 1$. Show that f has exactly one fixed point in the unit disk $|z| < 1$.

2. Find the number of zeros of $f(z) = z^4 + 3iz^2 + z - 2 + i$ in the upper half-plane.

3. Find the number of zeros of $f(z) = z^9 + 3z^8 - 6z^4 - 1$ in each of the regions $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$.

4. Find a conformal mapping of the quarter circle $D = \{z = x + iy : |z| < 1 \, , \, x > 0 \, , \, y > 0\}$ onto the upper half plane. This mapping cannot be a linear fractional mapping: why?

5. Evaluate
$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} \, dx$$
Be sure to fully justify your solution.

6. (a) Show that the mapping $w = z + \frac{1}{z}$ maps the unit circle $|z| = 1$ to the segment $[-2, 2]$ ($\{w = x + iy : -2 \leq x \leq 2 \, , y = 0\}$).
   (b) Use the mapping from part a) to determine the complex velocity potential $F(z)$ for a vertical flow around the segment from $[-2, 2]$ (i.e. $v_\infty = i$) and with no circulation ($c = 0$).
   (c) Find an equation (implicit or explicit) for the streamlines and sketch some of them.

7. Find the temperature distribution inside $D = \{z = x + iy : |z| < 1 \, , y > 0\}$, if the temperature is T_1 along the bottom and T_2 along the semi-circle. Sketch some isotherms.

8. Find a linear fractional mapping which has $z = i$ as its sole fixed point in the extended complex plane (i.e. $z = \infty$ is not fixed). **Note:** this question is a bonus for students of 4020 worth up to an additional 5%. It is required for students of 5020.

Comprehensive Questions

9. Prove the following:

   Theorem 1 (Rouché's) *Let C denote a simple closed coutour and suppose that*

   - *two functions $f(z)$ and $g(z)$ are analytic inside and on C;*
   - *$|f(z)| > |g(z)|$ at each point on C.*

   Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.

10. Show that a linear fractional transform which maps the unit circle $|z| = 1$ onto itself must have the form
$$f(z) = a(z - b)/(1 - \bar{b}z) \, ,$$
with $|a| = 1$ and $|b| \neq 1$ or $f(z) = \frac{a}{z}$ with $|a| = 1$.