Math 4020/5020 - Analytic Functions
Homework #1 Solutions

1. Determine the different values of the following logarithms:

$\log(1)$ $\qquad$ $\log(-1/2)$ $\qquad$ $\log(-1+i)$ $\qquad$ $\log\left(\frac{a-ib}{a+ib}\right)$

In the following, $k \in \mathbb{Z}$.

(a) $\log(1) = i(2k\pi)$
(b) $\log(-1/2) = \log(1/2) + i(\pi + 2\pi k)$
(c) For this case we note that if $w = a + ib$ than

$$\log\left(\frac{a-ib}{a+ib}\right) = \log\left(\frac{\bar{w}}{w}\right)$$

If we write $w = re^{i\theta}$ then $\frac{\bar{w}}{w} = e^{-2\theta}$ where $\theta = \mathrm{Arctan}(b/a)$ if $a > 0$ and $\theta = \frac{\pi}{2}$ if $a = 0$. Then $\log\left(\frac{a-ib}{a+ib}\right) = i(\theta + 2k\pi)$

2. Determine all the roots of,

$\sin(z) = i$ $\qquad$ $\cos(z) = 2$ $\qquad$ $\cot(z) = 1 + i$

(a) $\sin(z) = i$ implies

$$\begin{aligned} \frac{e^{iz} - e^{-iz}}{2i} &= i\,, \\ e^{2iz} + 2e^{iz} - 1 &= 0\,, \\ e^{iz} &= -1 \pm \sqrt{2}\,. \end{aligned}$$

We now note that $-1 + \sqrt{2} > 0$ and $-1 - \sqrt{2} < 0$ so we will get the following two sets of solutions

$$z = -i(\ln(-1 + \sqrt{2}) + i2k\pi)$$

and

$$z = -i(\ln(1 + \sqrt{2}) + i(2k\pi + \pi)$$

(b) $\cos(z) = 2$ implies

$$\begin{aligned} \frac{e^{iz} + e^{-iz}}{2} &= 2\,, \\ e^{2iz} - 4e^{iz} + 1 &= 0\,, \\ e^{iz} &= 2 \pm \sqrt{3}\,. \end{aligned}$$

In this case, both roots are positive, so we have

$$z = -i(\ln(2 \pm \sqrt{3}) + i2k\pi)$$

(c) $\cot(z) = 1 + i$ implies

$$\begin{aligned} i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} &= 1 + i\,, \\ e^{2iz} + 1 &= (1 - i)(e^{2iz} - 1)\,, \\ e^{2iz} &= 1 + 2i = \sqrt{5}e^{i\theta}\,, \text{ where } \theta = \text{Arctan}(2)\,, \\ e^{iz} &= \pm 5^{1/4}e^{i\theta/2}\,. \end{aligned}$$

Then $z = -\frac{i}{4}\ln(5) + \frac{\theta}{2} + k\pi$. Note the $k\pi$ takes care of the $\pm$ cases.

3. Assume $w = f(z) = u(x, y) + iv(x, y)$ is analytic in some domain D. Show the sets of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ intersect orthogonally.

   This question was done well, so I will be brief. $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$. So $\nabla u \cdot \nabla v = u_x v_x + u_y v_y$ and by the Cauchy-Riemann equations this is 0, so the curves are orthogonal.

4. Show the mapping $w = z^{\alpha+i\beta}$ maps the rays $\arg(z) = c_1$ and the circles $|z| = c_2$ into mutually orthogonal logarithmic spirals.

   We let $z = re^{i\theta}$, then

$$\begin{aligned} w &= e^{(\alpha+i\beta)(\ln(r)+i\theta)}\,, \\ &= e^{\alpha\ln(r)-\beta\theta}e^{i(\alpha\theta+\beta r)}\,. \end{aligned}$$

   If we let $w = Re^{i\Theta}$, then in the w-plane, $R = e^{\alpha\ln(r)-\beta\theta}$ and $\Theta = \alpha\theta + \beta\ln(r)$.

   $\boldsymbol{\theta = c_1}$ In this case, we can solve for $\ln(r) = \frac{1}{\beta}(\Theta - \alpha c_1)$ Then

$$\begin{aligned} R &= e^{\frac{\alpha}{\beta}(\Theta-\alpha c_1)-\beta c_1}\,, \\ &= e^{-\frac{\alpha^2 c_1}{\beta}-\beta c_1}e^{\frac{\alpha}{\beta}\Theta}\,. \end{aligned}$$

   Which defines a logarithmic spiral.

   $\boldsymbol{r = c_2}$ In this case, $\theta = \frac{1}{\alpha}(\Theta - \beta\ln(c_2))$, so

$$\begin{aligned} R &= e^{-\frac{\beta}{\alpha}(\Theta-\beta\ln(c_2))+\alpha\ln(c_1)}\,, \\ R &= e^{\frac{\beta^2}{\alpha}\ln(c_1)+\alpha\ln(c_2)}e^{-\frac{\beta}{\alpha}\Theta}\,. \end{aligned}$$

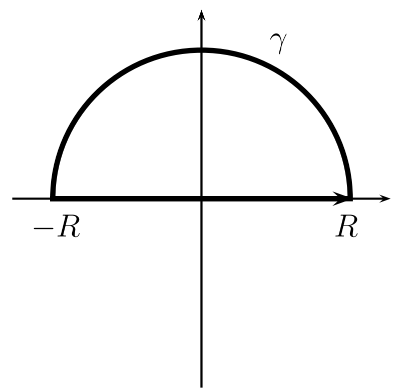
   Which also defines a logarithmic spiral.

   Now are the two sets of curves orthogonal? There are many ways to see this, here is one that no one tried. If you are given a polar curve in the form $r = f(\theta)$, the slope of the curve at the point θ is given by $\frac{f'(\theta)\sin(\theta)+f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta)-f(\theta)\sin(\theta)}$. This formula is derived by considering the polar curve as the parametric curve $(f(\theta)\cos(\theta), f(\theta)\sin(\theta))$. With this formula, we get the two slopes to be

$$\begin{aligned} m_1 &= \frac{\frac{\alpha}{\beta}\sin(\Theta) + \cos(\Theta)}{\frac{\alpha}{\beta}\cos(\Theta) - \sin(\Theta)}\,, \\ m_2 &= \frac{-\frac{\beta}{\alpha}\sin(\Theta) + \cos(\Theta)}{-\frac{\beta}{\alpha}\cos(\Theta) - \sin(\Theta)}\,. \end{aligned}$$

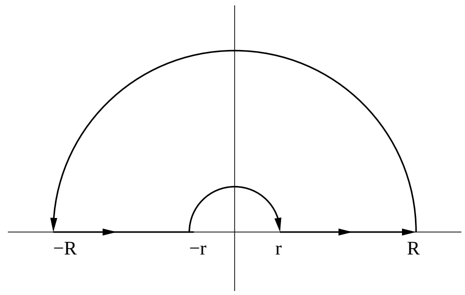
   With a little manipulation it is clear that $m_1 = -1/m_2$

5. Use residues to calculate
$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx\,.$$
To calculate this integral, we consider the following path:



Since the degree of the denominator is 2 greater then the degree of the numerator, we get,
$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = \int_{\gamma} \frac{z}{(z^2+1)(z^2+2z+2)}\,dz = 2\pi\Sigma\,\mathrm{Res}(f, z_i)$$
Where z_1 are the poles inside of γ. There are two simple poles in γ, one at $z_1 = i$ and one at $z = -1+i$. A simple calculation yields
$$\mathrm{Res}(f, i) = \frac{1-2i}{10}\,,$$
$$\mathrm{Res}(f, -1+i) = \frac{-1+3i}{10}\,.$$
Putting it all together gives
$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = 2\pi i\,\frac{i}{10}\,,$$
$$= -\frac{\pi}{5}\,.$$

6. Show that
$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$
by integrating an appropriate branch of the multi-valued function
$$f(z) = \frac{e^{(-1/2)\ln(z)}}{z^2+1}$$
over the contour below as $R \to \infty$ and $r \to 0$.



We Choose the branch of log with the negative complex axis removed. We label the large arc C_R and the small arc C_r and look at the integral around these sections of γ.

On C_R, $z = Re^{i\theta}$ and,

$$\begin{aligned}\int_{C_R} f(z)\,dz &= \int_0^{\pi} \frac{e^{(-1/2)(\ln(R)+i\theta)}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta\,,\\ &\leq \pi\frac{R}{\sqrt{R}(R^2-1)}\,,\\ &\to 0 \text{ as } R\to\infty\,.\end{aligned}$$

On C_r, we have

$$\begin{aligned}\int_{C_r} f(z)\,dz &= \int_0^{\pi} \frac{e^{(-1/2)(\ln(r)+i\theta}}{r^2e^{2i\theta}+1} ire^{i\theta}\,d\theta\,,\\ &\leq \pi\frac{r}{\sqrt{r}(1-r^2)}\,,\\ &\to 0 \text{ as } r\to 0\,.\end{aligned}$$

Now the only pole inside of γ is at $z = i$. This pole is simple and we calculate the residue as

$$\operatorname{Res}(f,i) = \frac{e^{-i\frac{\pi}{4}}}{2i} = \frac{1-i}{2\sqrt{2}i}$$

So, as $r\to 0$ and $R\to\infty$, we have:

$$\begin{aligned}\int_0^{\infty}\frac{e^{(-1/2)\ln(z)}}{z^2+1}\,dz + \int_{-\infty}^{0}\frac{e^{(-1/2)\ln(z)}}{z^2+1}\,dz &= 2\pi i\left(\frac{1-i}{2\sqrt{2}i}\right)\,,\\ \int_0^{\infty}\frac{dx}{\sqrt{x}(x^2+1)} - i\int_0^{\infty}\frac{dx}{\sqrt{x}(x^2+1)} &= \frac{\pi}{\sqrt{2}}(1-i)\,,\\ \int_0^{\infty}\frac{dx}{\sqrt{x}(x^2+1)} &= \frac{\pi}{\sqrt{2}}\,.\end{aligned}$$