Math 4020/5020 - Analytic Functions

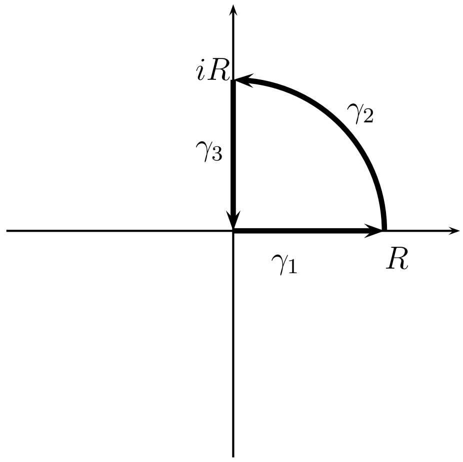
Homework #2 Solutions

1. Find the number of zeros of the polynomial f(z) in the first quadrant, where

   (a) $f(z) = z^3 - 3z + 6$

   (b) $f(z) = z^9 + 5z^2 + 3$

   For both problems, we will consider the following path γ,



   (a) Along γ_1, z is real and equal to x. Thus $\text{Arg}(f(z)) = 0$ on this segment. We must also ensure that $f(x) \neq 0$ on this segment. We note that the only critical point for $f(x)$ in $x \geq 0$ is at $x = 1$. At this point $f''(1) = 6 > 0$, so $x = 1$ is a local minimum. Hence on the positive real axis we have $f(x) > f(1) = 4 > 0$.
   Along γ_2, $z = Re^{i\theta}$, so $f(z) = R^3 e^{3i\theta}\left(1 + O(R^{-2})\right)$. So along γ_2, $\text{Arg}(f(z))$ goes from 0 to approximately $\frac{3\pi}{2}$.
   Finally along γ_3, $z = iy$, so $f(z) = -iy^3 - 3iy + 3$. Since $Re(f(z)) = 3 > 0$ and $Im(f(z)) = -y^3 - 3y \leq 0$, $f(z)$ will remain in the fourth quadrant along γ_3, so the total change in argument is 2π and thus $f(z)$ has one zero in the first quadrant.

   (b) It is clear that $f(z) > 0$ on γ_1, so we don't have to worry about any zeros there. Along this segment, $\text{Arg}(f(z)) = 0$
   Along γ_2, $z = Re^{i\theta}$ and so $f(z) = R^9 e^{9i\theta}\left(1 + O(R^{-7}\right)$, so $\text{Arg}(f(z))$ goes from 0 to approximately $\frac{9\pi}{2}$.
   On γ_3, $f(z) = iy^9 - 5y^2 + 3$. $Im(f(z)) = 0$ only at $y = 0$ and at this point $f(0) = 3 \neq 0$, so $f(z)$ does not have any zeros on this segment. As well, $Im(f(z)) > 0$ on this segment, so it is not possible for $f(z)$ to loop around the origin again and the total change in argument of $f(z)$ must be 4π, so there are two zeros in the first quadrant.

2. Find the number of zeros of $f(z)$ in the given annulus:

   (a) $p(z) = z^4 - 2z - 2$ in $\frac{1}{2} < |z| < \frac{3}{2}$,
   First let $f(z) = -2$ and $g(z) = z^4 - 2z$. So if $z = \frac{1}{2}e^{i\theta}$, then $|g(z)| \leq 1\frac{1}{16} < |f(z)|$. Thus $f(z)$ and $f(z) + g(z)$ have the same number of zeros in $|z| < 2$ namely 0.
   Now let $f(z) = z^4$ and $g(z) = -2z - 2$. If $z = \frac{3}{2}e^{i\theta}$ then $|g(z)| \leq 3 + 2 < \frac{81}{16} = |f(z)|$, so $f(z)$ and $p(z) = f(z) + g(z)$ have the same number of zeros in $|z| < \frac{3}{2}$ namely 4. Thus, all four zeros of p are in $\frac{1}{2} < |z| < \frac{3}{2}$.

   (b) $p(z) = ze^z - \frac{1}{4}$ in $0 < |z| < 2$.

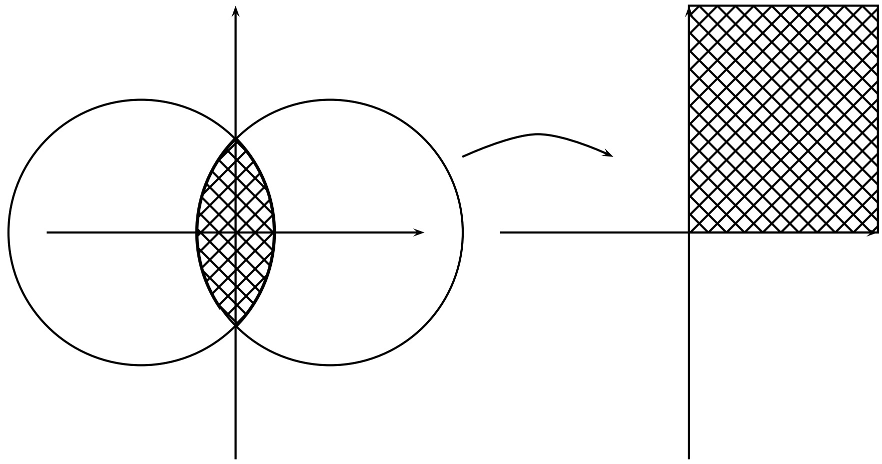
Since $f(0) \neq 0$, this is the same as finding the number of zeros in $|z| < 2$. On $|z| = 2$, $|ze^z| = 2e^{Re(z)} > 2e^{-2} = 0.276\ldots > \frac{1}{4}$. So $p(z)$ and ze^z have the same number of zeros in $|z| < 2$ namely 1.

3. Let f and g be analytic inside a simple closed curve γ and suppose that $f(z) \neq 0$ inside of γ. Show that if $|f(z)| \geq |g(z)|$ for all $z \in \gamma$ then $|f(z)| \geq |g(z)|$ for all z inside of γ. Give an example to show that the assumption that $f(z) \neq 0$ inside of γ is necessary. (Note: you will have to consider what can happen if $f(z) = 0$ on γ.)
   Since f and g are analytic inside and on γ and $f(z) \neq 0$ inside of γ, then $h(z) = \frac{g(z)}{f(z)}$ is analytic inside of γ. It is also analytic on γ, since the only place h may not be analytic is at a point z_0 where $f(z_0) = 0$. Now $0 = |f(z_0)| \geq |g(z_0)|$, so $g(z_0) = 0$ as well.
   If z_0 is an order 1 root of f then h will have a removable singularity and we can make h analytic. If f has a second order root, we can apply the same argument to $f(z)/(z - z_0)$ and $g(z)/(z - z_0)$. We can then use induction to show that h is analytic for a root of arbitrary order.
   Since $h(z)$ is analytic in and on γ, its maximum value will occur on γ. On γ $|f| \geq |g|$, so $|h| \leq 1$ on γ. So $|h| \leq 1$ inside of γ as well. Then $|f| > |g|$ inside of γ.
   To show that $f(z) \neq 0$ is needed consider $f(z) = z$ and $g(z) = 1$ and γ the unit circle. On γ, $|f(z)| = |g(z)| = 1$, so $|f(z)| \geq |g(z)|$ on γ, but in γ, $|f(z)| < 1 = |g(z)|$.

4. Find a linear fractional transform that maps:
   (a) the circle $|z| = 1$ onto the line $Re((1+i)w) = 0$.
   If $w = x + iy$ then $Re((1+i)w) = x - y$, so the line is given by $x = y$ in the w-plane. So we choose to map $(-1, i, 1)$ to $(0, 1+i, \infty)$. The LFT must have the form $w = \alpha\frac{z+1}{z-1}$. Then $w(i) = \alpha\frac{i+1}{i-1} = -\alpha i = 1 + i$. So $\alpha = i - 1$ and $w = (i-1)\frac{z+1}{z-1}$.
   (b) the circle $|z| = 1$ onto the circle $|w - 1| = 1$.
   We can do this one by inspection as all we need is a translation. Thus, $w = z + 1$ will work.
   (c) the real axis onto the line $Re(w) = 1/2$.
   We can just rotate and translate. So, $w = iz + \frac{1}{2}$ will work.

5. Find the image of the following sets under $w = \frac{z-i}{z+i}$:
   (a) the real axis.
   We see where three points on the real axis are mapped. Note $(0, 1, \infty)$ are mapped to $(-1, -i, 1)$. These 3 lie on the circle $|z| = 1$.
   (b) the circle $|z| = 1$.
   Again we pick three points and see where they are mapped to. Now $(-i, 1, i) \to (\infty, -i, 0)$. So the circle must be mapped to the imaginary axis.
   (c) the imaginary axis.
   We note $(0, i, \infty) \to (-1, 0, 1)$. So the image must be the real axis.

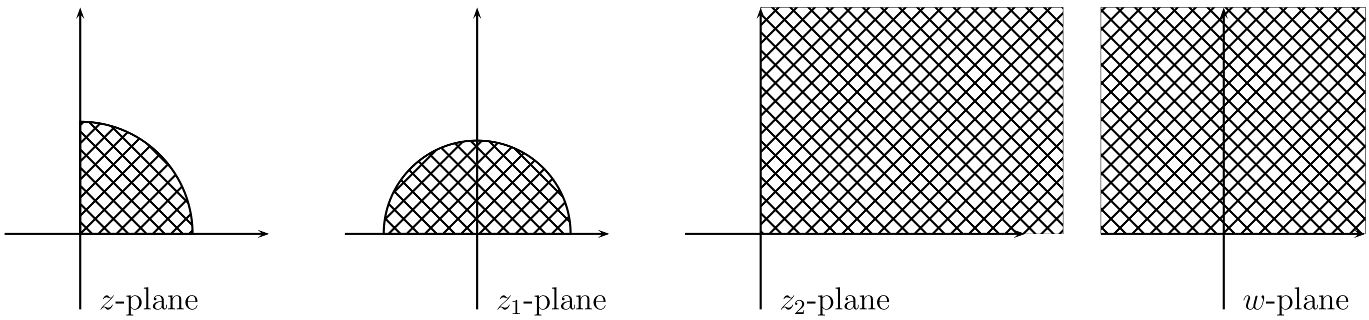
6. (a) Find a conformal mapping of the region $D = \{z : |z-1| < \sqrt{2}, |z+1| < \sqrt{2}\}$ onto the open first quadrant.
   Here is a picture of the regions.



We first note that the two circles intersect at right angles. By conformality, the images will intersect at right angles. So if we find a LFT which maps $(-i, i, \sqrt{2}-1) \to (0, \infty, 1)$, we should have the required mapping. The mapping will have the form $f(z) = \alpha\frac{z+i}{z-i}$. Let $\beta = \sqrt{2} - 1$ and $f(\beta) = \alpha\frac{\beta+i}{\beta-i} = 1$, so $\alpha = \frac{\beta-i}{\beta+i}$ and the mapping is $f(z) = \frac{\beta-i}{\beta+i}\frac{z+i}{z-i}$.

(b) Find a conformal mapping of D onto the upper half plane. This can't be a linear fractional mapping. Why?
To map D onto the upper half plane, all we need to do is to square the previous mapping, so $f(z) = \left(\frac{\beta-i}{\beta+i}\frac{z+i}{z-i}\right)^2$. We couldn't accomplish this with a LFT since the pre-image has boundaries with 2 right angles. A LFT would have to preserve at least one of these angles and there are no right angles on the boundary of the image.

7. Find a conformal mapping of the quarter circle $D = \{z = x + iy : |z| < 1, x > 0, y > 0\}$ onto the upper half plane. This can't be a linear fractional mapping either. Why?
There are many ways to accomplish this, I will use a series of mappings as illustrated below.



From the picture we construct the following maps:

$$z_1 = z^2\,,$$
$$z_2 = \frac{1+z_1}{1-z_1}\,,$$
$$w = z_2^2\,.$$

The final map is given by,

$$w = \left(\frac{1+z^2}{1-z^2}\right)^2 .$$

Again this mapping cannot be done with just a linear fraction map. The boundary of the first domain has 3 right angles and the boundary of the final domain has no right angles.