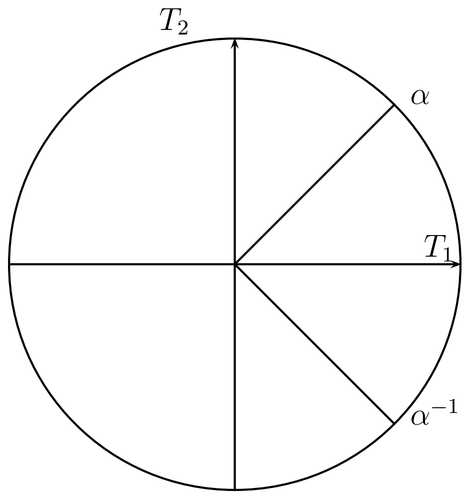
Math 4020/5020 - Analytic Functions

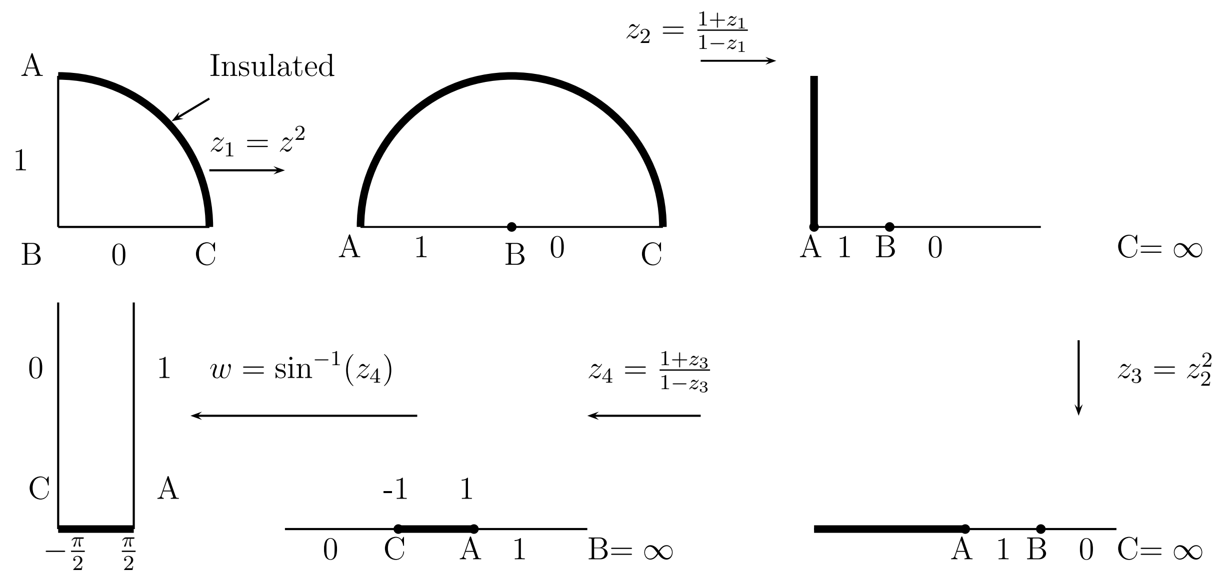
Solutions

1. An arc of θ_0 radians in a circle is kept at temperature T_1 wile the remainder of the circle is kept at T_2. By mapping into the upper half plane, find the temperature distribution inside the circle.
   Let the arc to be kept at T_1 be $|\theta| < \frac{\theta_0}{2}$ and $\alpha = e^{i\theta_0/2}$.

   We map $|z| < 1$ to the upper half plane with a linear fractional transform by sending
   $$\alpha^{-1} \to 0\,,$$
   $$1 \to 1\,,$$
   $$\alpha \to \infty\,.$$
   so,
   $$f(z) = \beta \frac{z - \alpha^{-1}}{z - \alpha}$$
   where $f(1) = \frac{\beta}{\alpha}\frac{\alpha-1}{1-\alpha} = 1$, so $\beta = -\alpha$. We can then write,
   $$f(z) = \frac{1 - \alpha z}{z - \alpha}$$
   In the w-plane, a suitable function is,
   $$\psi(w) = T_1 + (T_2 - T_1)\frac{1}{\pi}\operatorname{Arg}(w)$$
   So the solution is the z-plane is given by
   $$T(z) = T_1 + (T_2 - T_1)\operatorname{Arg}\left(\frac{1 - \alpha z}{z - \alpha}\right)\,,$$
   where $\alpha = e^{i\theta_0/2}$.

2. Let D be the quartercircle $\{z : |z| < 1, x > 0, y > 0\}$. Find the electrostatic potential ϕ in D (harmonic in D) with the following boundary conditions: $\phi = 0$ on the real axis, $\phi = 1$ on the imaginary axis and $\frac{\partial \phi}{\partial n} = 0$ on the circular part (no flux).
   We consider the following sequence of maps



The mapping $z_4 = \frac{1+z_3}{1-z_3}$ is needed to move the insulated section on the boundary to the interval $[-1, 1]$. After working through the various maps we get $z_4 = \frac{1+z^4}{-2z^2}$ and then the solution is

$$T = \frac{1}{2} + \frac{1}{\pi} \sin^{-1} \left(\frac{1+z^4}{-2z^2} \right) .$$

3. Find a stream function and velocity potential function for flows with the following velocity fields:

   (a) $f = \bar{z}$.
   The complex potential G satisfies $G'(z) = z$, so $G(z) = \frac{z^2}{2}$. The velocity potential is then $\Phi = xy$ and the stream function is $\Psi = (x^2 - y^2)/2$ where $z = x + iy$.

   (b) $f = \sin(\bar{z})$.
   We note that $\overline{\sin(\overline{z})} = \sin(z)$ (use the definition $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$), then we have the complex potential satisfies $G'(z) = \sin(z)$, so $G(z) = -\cos(z)$ and here $\Phi = -\cos(x)\cosh(y)$ and $\Psi = \sin(x)\sinh(y)$.

   (c) $f = \frac{\bar{z}-1}{\bar{z}+1}$
   Here G satisfies $G'(z) = \frac{z-1}{z+1}$, so $G(z) = z - 2\log(z+1)$ and $\Phi = x - 2\log(\sqrt{(x+1)^2 + y^2})$ and $\Psi = y - 2\,\mathrm{Arg}(z+1)$ or $y - 2\arctan(y/x+1)$.

   Note: If $f = u + iv$ the velocity of the flow will be given by the vector (u, v).

4. Using the fact that the Joukowski mapping $w = z + \frac{1}{z}$ maps circles $|z| = r > 1$ to ellipses, find the complex velocity potential for flow past an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ with no circulation.
   We want the flow with $c = 0$ around $K = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$. So, we to find a map which sends the ellipse to the unit circle. We may then use the flow around the unit circle with no circulation.

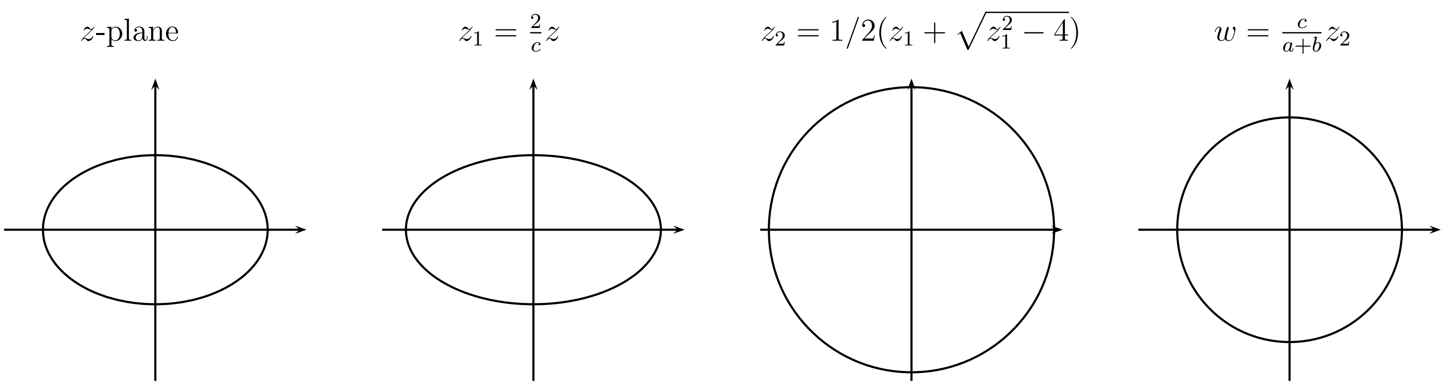
We know that $w = z + \frac{1}{z}$ maps the circle $|z| = r$ to the ellipse

$$\left(\frac{u}{r+\frac{1}{r}}\right)^2 + \left(\frac{v}{r-\frac{1}{r}}\right)^2 = 1$$

First we note that the distance between the foci squared $c^2 = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4$ for this ellipse. So before we can use this transform, we must scale our ellipse to have $c = 2$ So if $a > b > 0$, then we let $c = \sqrt{a^2 - b^2}$ and we can set $r > 1$ such that,

$$\begin{aligned} \frac{2a}{c} &= r + \frac{1}{r}\,, \\ \frac{2b}{c} &= r - \frac{1}{r}. \end{aligned}$$

So $r = \frac{a+b}{c}$ and $\frac{1}{r} = \frac{a-b}{c}$. This is consistent since $c^2 = a^2 - b^2$. With this choice of r, $w = z + \frac{1}{z}$ will map $|z| = r$ to the correct ellipse.
We will need a mapping going the other way. If we solve for z, we get $z = \frac{w \pm \sqrt{w^2-4}}{2}$. We have two roots since $w = f(z)$ will always map z and $\frac{1}{z}$ to the same point. Since one of z and $\frac{1}{z}$ will be inside the unit circle and one will be outside, the mapping is 1-1 on the domain of interest. If we choose the correct branch of $\sqrt{w^2-4}$, we can construct a consistent inverse. We can do this by writing $\sqrt{w^2-4} = (w-2)^{1/2}(w+2)^{1/2}$ and restricting the arguments of $w \pm 2$ to $0 < \operatorname{Arg}(w \pm 2) < 2\pi$. With this choice, $\sqrt{w^2-4}$ is defined and consistent except for a branch cut joining -2 to 2. The inverse of $f(z)$ is then $z = \frac{w+\sqrt{w^2-4}}{2}$.
We are now ready to construct our mapping, $h(z)$, from the ellipse to the unit circle. We will use several stages to get the scaling right.



Putting it all together, we have

$$\begin{aligned} w &= \frac{c}{a+b} z_2\,, \\ &= \frac{c}{2(a+b)}\left(z_1 + \sqrt{z_1^2 - 4}\right), \\ &= \frac{c}{2(a+b)}\left(\frac{2}{c}z + \sqrt{\frac{4}{c^2}z^2 - 4}\right), \\ &= \frac{z + \sqrt{z^2 - c^2}}{a+b} \end{aligned}$$

Now to find a flow with no circulation, we us $G(w) = \lambda w + \frac{\bar{\lambda}}{w}$ as the flow around the unit disc. Now we just use our mapping to the z-plane to get

$$G(z) = \lambda \frac{z + \sqrt{z^2 - c^2}}{a + b} + \frac{\bar{\lambda}(a + b)}{z + \sqrt{z^2 - c^2}}$$

We can further simplify to get,

$$G(z) = \lambda \frac{z + \sqrt{z^2 - c^2}}{a + b} + \frac{\bar{\lambda}(a + b)(z - \sqrt{z^2 - c^2})}{c^2}$$

5. Let $w = f(z)$ be a non-constant analytic function mapping a domain D to the domain E. Suppose tht ψ is a smooth function on E and that ϕ is defined on D by $\phi(z) = \psi(f(z))$. Show that $\phi_{xx} + \phi_{yy} = |f'(z)|^2 (\psi_{uu} + \psi_{vv})$.
   $\phi(x, y) = \psi(u(x, y), v(x, y))$, where $f = u + iv$. So

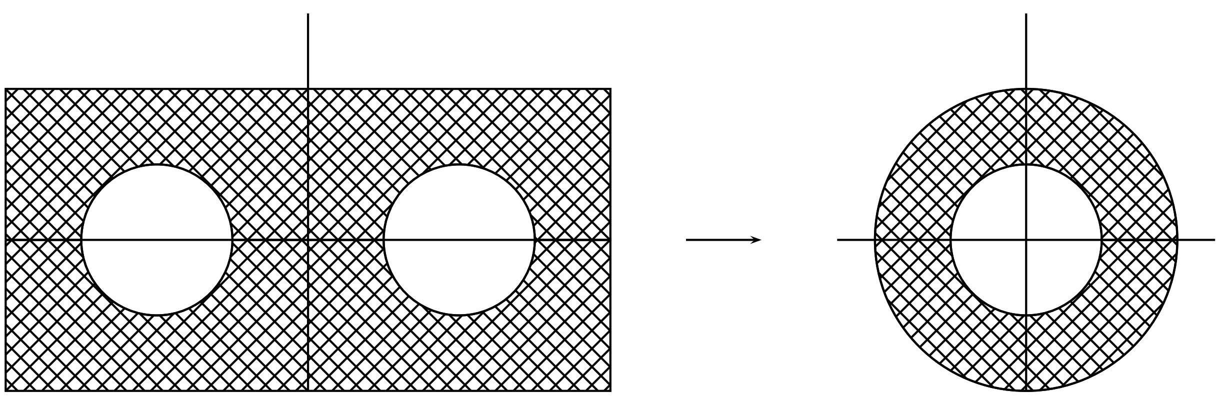
$$\begin{aligned}
\phi_x &= \psi_u u_x + \psi_v v_x\,, \\
\phi_y &= \psi_u u_y + \psi_v v_y\,, \\
\phi_{xx} &= \psi_{uu} u_x^2 + \psi_{vv} v_x^2 + \psi_u u_{xx} + \psi_v v_{xx}\,, \\
\phi_{yy} &= \psi_{uu} u_y^2 + \psi_{vv} v_y^2 + \psi_u u_{yy} + \psi_v v_{yy}\,.
\end{aligned}$$

   Now using $\Delta u = \Delta v = 0$, we have

$$\Delta\phi = \psi_{uu}(u_x^2 + y_y^2) + \psi_{vv}(v_x^2 + v_y^2)\,.$$

   But $f'(z) = u_x + iv_x = u_x + -iu_y = v_y + iv_x$, and $|f'(z)|^2 = u_x^2 + u_y^2 = v_y^2 + v_x^2$, which gives us the required result.

6. Find the electrostatic potential between the two cylinders with cross-sections $\{z : |z - 2| = 1\}$ and $\{z : |z + 2| = 1\}$ if the first cylinder has charge Q_1 and the second Q_2. This question is a bonus for those enrolled in math4040.
   We must find a map between the following two regions.



First we translate the left circle to the unit circle centered at the origin with $z_1 = z + 2$. Now a transform of the $w = \frac{z_1 - a}{1 - az_1}$ with $|a| > 1$ will fix the unit circle and map the circle on the right to another circle. We choose a so that $z_1 = 3$ gets mapped to $w = R$ and $z_1 = 5$ gets mapped to $w = -R$. Then the circles will be concentric. $w(3) = -w(5) = R$ reduces to

$$\frac{3 - a}{1 - 3a} = -\frac{5 - a}{1 - 5a}\,,$$

or $a = 2 + \sqrt{3}$ $(2 - \sqrt{3} < 0)$. Then $R = 7 - 4\sqrt{3}$. In the w-plane the solution is given

$$\psi(w) = V_2 + \frac{V_2 - V_1}{\log(r)} \log |w| \, .$$

Finally in the z-plane we have

$$\psi(w) = V_2 + \frac{V_2 - V_1}{\log(r)} \log \left| \frac{z + 2 - a}{1 - a(z + 2)} \right| \, .$$