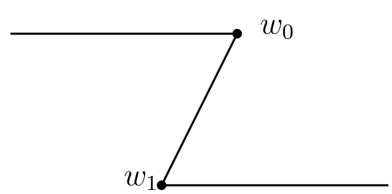
Math 4020/5020 - Analytic Functions

Homework #4 Due March 27th

1. Find the Schwartz-Christoffel transformation which maps the upper half-plane onto the triangle with vertices $\{0, i, 1\}$. Choose $x_1 = -1$, $x_2 = 1$.
   We can write the mapping as
   $$f(z) = A\int_1^z (\zeta+1)^{-3/4}(\zeta-1)^{-3/4}d\zeta + B$$
   or
   $$f(z) = A\int_{-1}^z \frac{d\zeta}{(\zeta^2-1)^{3/4}} + B$$
   We can just use $f(-1) = i$ to get $B = i$. If we let
   $$k = \int_{-1}^1 \frac{d\zeta}{(\zeta^2-1)^{3/4}}$$
   then we can solve for A as
   $$A = -ik$$
   . We note that although, we have an improper integral for k, we can still solve it. Using Maple, we get the approximation
   $$k \sim -3.708149355 - 3.708149355i$$
   .

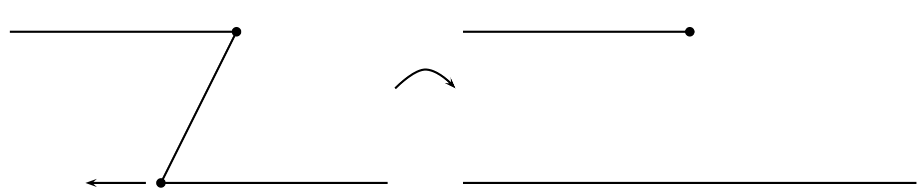
2. Find a Schwartz-Christoffel transformation which maps the upper half-plane onto the domain $D = \{z : 0 < Arg(z) < 4\pi/3\}$
   We can choose $x_1 = w_1 = 0$. Here the exterior angel is $-\frac{\pi}{3}$, so $\alpha_1 = \frac{1}{3}$ and the transform $f(z)$ must satisfy $f'(z) = A(z-0)^{1/3}$, so $f(z) = A\int_0^z w^{1/3}\,dw$. Thus $f(z) = \frac{3A}{4}z^{4/3}$. Since we only have the one restriction, we may choose $A = 4/3$, to end with $f(z) = z^{4/3}$.

3. (a) Use Schwarz-Christoffel to find a mapping of the upper-half plane into the domain above the "z"-type boundary as shown below. Note: only write $f'(z)$ down, you don't need to solve for f.

   We take $x_0 = -1$ and $x_1 = 0$. Then $f'(z) = A(z+1)^{\theta_0/\pi}z^{\theta_1/\pi}$, where θ_i are the exterior angles shown.

   (b) Take the limit as indicated in the figure below. Find $f(z)$ explicitly in this case.

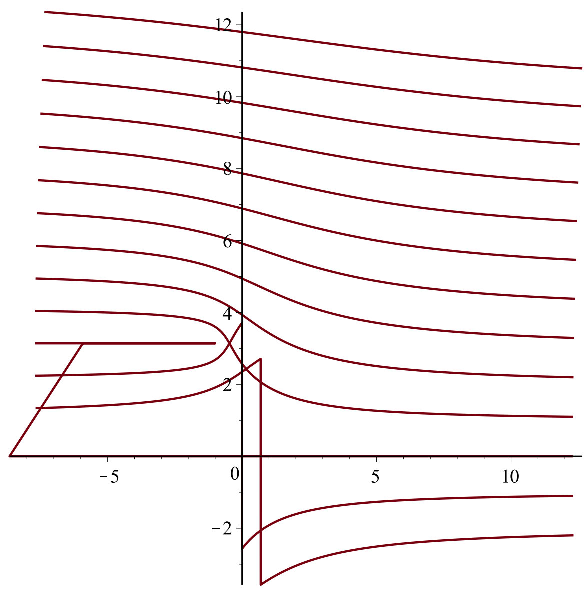
As $R \to \infty$, $\theta_0 \to -\pi$ and $\theta_1 \to \pi$. So we get $f'(z) = A(z+1)z^{-1}$. If we set $w_0 = -1+i\pi$, then for any $x < -1$ we have

$$\pi = \Im(f(x)) = \Im(Ax + \log|x| + i\pi + B)\,,$$

This implies $x\Im(A) + \Im(B) = 0$ for any $x \le -1$, thus A and B must be real. Now $-1 + i\pi = f(-1) = A(-1 + i\pi) + B$, which implies $A = 1$ and $B = 0$. So,

$$f(z) = z + \log(z)\,.$$

(c) Use a computer to sketch a few streamlines for the domain in part 3b Using my maple code posted in class I got the following:



4. Apply the ideas of Step 1 of the Riemann mapping theorem to devise invertible, conformal mappings of the following simply connected domains to domains that lie within the unit disc $|w| < 1$ and include the point $w = 0$.

   (a) The strip $0 < \Re(z) < 100$.
   The domain $\{z||z-110| < 5\}$ is completely outside of the strip. So the mapping $w = \frac{5}{z-110}$ will map the strip to inside the unit disc.

   (b) The double cut plane obtained by deleting the rays $(-\infty, -1]$ and $[-1, \infty)$ from $\mathbb{C}$.
   We can't use this type of map on this domain directly. We must first open up one of the slits. So we let $z_1 = \sqrt{z+1} = \sqrt{|z+1|}e^{i\theta/2}$ where θ is the argument of $z - 1$ and is in $(\pi, -\pi)$. Then the map $w = \frac{1}{z_1+3}$ will map the domain to the inside of the unit disc.

5. Show that the mapping

$$z = w\frac{\lambda - w}{1 - \lambda w}\,,$$

from $w \to z$ is a contraction if $\lambda = \frac{2\sqrt{r}}{1+r} < 1$, whenever $|w| < 1$. This question is a bonus students in 4020.
This is a little more tricky than I had thought. You need to be careful and it requires the definition of λ. So recall

$$\lambda = \frac{2\sqrt{r}}{1+r}\,,$$

with $r < 1$ and $|w| \leq r$. We consider the worst case scenario and let $w = re^{i\theta}$. Then

$$\left|\frac{\lambda - w}{1 - \lambda w}\right|^2 = \frac{\lambda - re^{i\theta}}{1 - \lambda re^{i\theta}} \frac{\lambda - re^{-i\theta}}{1 - \lambda re^{-i\theta}}, \tag{1}$$

$$= \frac{\lambda^2 - \lambda r(e^{i\theta} + e^{-i\theta}) + r^2}{1 - \lambda r(e^{i\theta} + e^{-i\theta})}, \tag{2}$$

$$= \frac{\lambda^2 - 2\lambda r\cos(\theta) + r^2}{1 - 2\lambda r\cos(\theta) + \lambda^2 r^2}. \tag{3}$$

We plug in for λ to get

$$\left|\frac{\lambda - w}{1 - \lambda w}\right|^2 = \frac{r\left(-4 + 4\sqrt{r}\cos(\theta) + 4\,r^{3/2}\cos(\theta) - r - 2\,r^2 - r^3\right)}{-1 - 2\,r - r^2 + 4\,r^{3/2}\cos(\theta) + 4\,r^{5/2}\cos(\theta) - 4\,r^3}.$$

The above function has 3 critical points in $[0, 2\pi]$ at $t = 0$, $t = \pi$ and $t = 2\pi$. At $t = 0$ and $t = 2\pi$, we have a local minimum and at $t = \pi$ we have a local maximum. At $t = \pi$, we have

$$\left|\frac{\lambda - w}{1 - \lambda w}\right|^2 = \frac{r\left(4 + 4\sqrt{r} + 4\,r^{3/2} + r + 2\,r^2 + r^3\right)}{1 + 2\,r + r^2 + 4\,r^{3/2} + 4\,r^{5/2} + 4\,r^3}.$$

For $0 \leq r \leq 1$, the above in a strictly increasing function of r and attains the value of 1 at $r = 1$.