Math 4220/5220 -Introduction to PDE's

Take Home Exam – Due April 17

1. Solve $xu_t + uu_x = 0$ with $u(x,0) = x$. **Hint:** change variables $y = x^2$.
2. Solve the wave equation in three dimensions:

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy} + u_{zz})\,, \\ u(\vec{x},0) &= 0\,, \\ u_t(\vec{x},0) &= x^2 + y^2 + z^2\,. \end{aligned}$$

   Hint: You should consider the radially symmetric Laplacian operator.
3. Find all three dimensional plane waves; that is, all solutions of the form $u(\vec{x},t) = f(\vec{k}\cdot\vec{x} - ct)$, where $\vec{k}$ is a fixed vector and f is a scalar function.
4. Find the Green's function for the first quadrant $\{(x,y)|x > 0\,,\ y > 0\}$. Use your answer to solve the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} = 0 \text{ in } Q\,, \\ u(0,y) = g(y)\,, \\ u(x,0) = h(x)\,. \end{aligned}$$

5. Solve the following heat equation:

$$\begin{gathered} u_t - u_{xx} = 0\,, \quad 0 < x < L\,, \quad t > 0\,, \\ u(x,0) = \phi(x)\,, \\ u(0,t) = a + ct\,, \quad u(L,t) = b + ct\,. \end{gathered}$$

   where $\phi(0) = a$ and $\phi(L) = b$.
6. Find $u(\pi,t)$ and $u(-\pi,t)$ for $t = 0\,, \frac{\pi}{2}\,, \pi\,, \frac{3\pi}{2}\,, 2\pi$ where u satisfies

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0\,, \quad x \in \mathbb{R}\,, \quad t > 0\,, \\ u(x,0) = \phi(x)\,, \\ \frac{\partial u}{\partial t}(x,0) = \psi(x)\,, \end{gathered}$$

   where

   (a)

$$\phi(x) = \begin{cases} \cos(x) & |x| < \frac{\pi}{2} \\ 0 & |x| \geq \frac{\pi}{2} \end{cases}$$

$$\psi(x) = 0\,.$$

   (b)

$$\psi(x) = \begin{cases} \cos(x) & |x| < \frac{\pi}{2} \\ 0 & |x| \geq \frac{\pi}{2} \end{cases}$$

$$\phi(x) = 0\,.$$

7. Let u satisfy,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0\,, \quad x \in \mathbb{R}\,, \quad t > 0\,,$$
$$u(x,0) = \phi(x)\,,$$
$$\frac{\partial u}{\partial t}(x,0) = \psi(x)\,,$$

where $\phi(x)$ and $\psi(x)$ have compact support (are identically 0 for $|x| > M$ for some M). Show

$$\int_{-\infty}^{\infty} \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right)\, dx$$

is constant in time (i.e. energy is conserved). (Hint: you may want to consider($\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)$)

8. (a) Let $U(x)$ be a continuous differentiable function and w_i be and approximation to $U(x_i)$ where $x_i = a + ih$ for some a, $h > 0$ and integer i. Use the Taylor series expansions of $U(x_i \pm h)$ and $U(x_i \pm 2h)$ to construct an approximation to $U''(x_i)$ in terms of w_{i-2}, w_{i-1}, w_i, w_{i+1} and w_{i+2} with error term of order $O(h^4)$(here w_i is the approximation to $U(x_i)$).

   (b) Use the result from a) to find a time stepping method for

$$u_{tt} = c^2 u_{xx}\,.$$
$$u(0,t) = u(1,t) = 0\,.$$

   You may assume that 4 previous time steps are already given. Denote the time step as k and the space step as h.

Note: Questions 7 and 3 are a bonus for students enrolled in 4220. The bonus questions may add an additional 5% to your mark.