Math 4220/5220 -Introduction to PDE's

Homework #2 Due Feb. 13^{th}

1. Use separation of variables to solve the non-homogeneous problem,
$$\phi_t = a^2\phi_{xx} + 1 \quad \text{for} \quad 0 < x < L\,,\, t > 0\,,$$
for $\phi(x,0) = 0$, $\phi(0,t) = t$, $\phi_x(L,t) = -c\phi(L,t)$ where $c > 0$ is a constant. (**Note:** you won't be able to solve for the eigenvalues exactly).

2. Show that the equation
$$u_t = u_{xx} + Q(x)\,, \quad 0 \leq x \leq L\,, \quad t > 0\,,$$
with the boundary conditions $u_x(0) = u_x(L) = 0$ has no equilibrium solution unless $\int_0^L Q(x)dx = 0$. In other words show an insulated bar, to which energy is being added (or subtracted), can not obtain a thermal equilibrium.

3. Find the solution, $u(x,y,t)$, to the following PDE:
$$u_t = k(u_{xx} + u_{yy})\,, \quad 0 \leq x \leq 1\,, \quad 0 \leq y \leq 1\,, t > 0\,,$$
with the boundary conditions $u_x(0,y,t) = u_x(1,y,t) = 0$, $u(x,0,t) = 0$, $u(x,1,t) = 1$ and initial conditions $u(x,y,0) = \phi(x,y)$. **Note: you may leave your answer in the form of an infinite series, but be sure to define all the terms of the series in terms of the given functions**.

4. Use the Rayleigh quotient (energy functional) to obtain an upper bound for the lowest eigenvalue of
   (a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(1) = 0$.
   (b) $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) + \phi(1) = 0$.

5. Consider the eigenvalue problem,
$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0\,,$$
subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

6. A rod occupies the interval $1 < x < 2$. The thermal conductivity depends on x in such a way that the temperature $\phi(x,t)$ satisfies the equation,
$$\phi_t = A^2(x^2\phi_x)_x$$
where A is a constant. For $\phi(1,t) = 0 = \phi(2,t)$ for $t > 0$ and $\phi(x,0) = f(x)$ for $1 < x < 2$, show that the appropriate eigenfunctions β_n are,
$$\beta_n(x) = \frac{1}{\sqrt{x}}\sin\left(\frac{\pi n \ln x}{\ln(2)}\right)$$
and work out the separation of variables solution of this problem.

For students enrolled in Math 5220 all questions are of equal value and required.

For students enrolled in Math 4220, question 6 is a bonus question worth up to an additional 5 %.