Math 4220/5220 -Introduction to PDE's
Homework #2 Solutions

1. Use separation of variables to solve the non-homogeneous problem,

$$\phi_t = a^2\phi_{xx} + 1 \quad \text{for} \quad 0 < x < L\,,\, t > 0\,,$$

for $\phi(x,0) = 0$, $\phi(0,t) = t$, $\phi_x(L,t) = -c\phi(L,t)$ where $c > 0$ is a constant. (**Note:** you won't be able to solve for the eigenvalues exactly).
The first step is to make the boundary conditions homogeneous. To do this, we write $\phi(x,t) = u(x,t) + z(x,t)$. We will want $u(0,t) = 0$ and $u_x(L,t) = -cu(L,t)$ in order for us to have an eigenvalue problem. First, $u(0,t) = \phi(0,t) - z(0,t) = t - z(0,t) = 0$ So we have $z(0,t) = t$. The other boundary condition is a little more complex. We will want the following:

$$\begin{aligned} u_x(L,t) &= -cu(L,t)\,, \\ \phi_x(L,t) - z_x(L,t) &= -c(\phi(L,t) - z(L,t))\,, \\ z_x(L,t) &= -cz(L,t)\,. \end{aligned}$$

So we have two conditions on z. We will guess the form of z to be, $z(x,t) = a(t) + b(t)x$. The first condition gives us $a(t) = t$. Now we apply the second condition,

$$\begin{aligned} b(t) &= -c(t + b(t)L)\,, \\ b(t) &= \frac{-ct}{1+cL}\,. \end{aligned}$$

So,

$$z(x,t) = t - \frac{ct}{1+cL}x\,.$$

We need to find the equation that u will satisfy.

$$\begin{aligned} \phi_t &= u_t + z_t = u_t + 1 - \frac{c}{1+cL}x\,, \\ \phi_{xx} &= u_{xx}\,, \\ \phi(x,0) &= u(x,0) = 0\,. \end{aligned}$$

So putting it all together, we must now solve

$$\begin{aligned} u_t &= a^2u_{xx} + \frac{c}{1+cL}x\,, \\ u(x,0) &= 0\,, \\ u(0,t) &= 0\,, \\ u_x(L,t) &= -cu(L,t)\,. \end{aligned}$$

We want to find the eigenvalues of the homogeneous problems so that we may guess the form of the solution. To that end we consider the eigenvalue problem,

$$\begin{aligned} X''(x) &= -\lambda^2X(x)\,, \\ X(0) = 0\,, &\quad X'(L) = -cX(L)\,. \end{aligned}$$

With this choice of eigenvalue problem the equation for the associated time component is,

$$T'(t) = -a^2\lambda^2 T(t)$$

We can solve the eigenvalue problem to get the eigenvalues are solutions to the transcendental equation,

$$\lambda_n = -c\tan(\lambda_n L)$$

The eigenfunctions are then given by $\sin(\lambda_n x)$. We now guess the form of the solution to be given by,

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(\lambda_n x)\,. \tag{1}$$

When we substitute this ansatz into the equation, we will wish to have all of the terms inside of the summation. So we will now write the function $W(x) = \frac{c}{1+cL}x$ as an eigenfunction expansion.

$$W(x) = \sum_{n=1}^{\infty} w_n \sin(\lambda_n x)\,,$$

where

$$w_n = \frac{2}{L}\int_0^L \frac{cx\sin(\lambda_n x)}{1+cL}\,dx = \frac{2c(\sin(\lambda_n L) - \lambda_n L\cos(\lambda_n L))}{\lambda_n^2 L(1+cL)}\,. \tag{2}$$

Plugging (2) and (1) into the equation we get,

$$\sum_{n=1}^{\infty}(\alpha_n(t)' + a^2\lambda_n^2\alpha_n(t) - w_n)\sin(\lambda_n x) = 0\,.$$

Thus, $\alpha_n(t)' + a^2\lambda_n^2\alpha_n(t) - w_n = 0$ for each n. We can now apply the initial conditions which give us $\alpha_n(0) = 0$ for all n. The solution for α_n is given by,

$$\alpha_n(t) = \frac{w_n}{a^2\lambda_n^2}\left(1 - e^{-a^2\lambda_n^2 t}\right)\,,$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{w_n}{a^2\lambda_n^2}\left(1 - e^{-a^2\lambda_n^2 t}\right)\sin(\lambda_n x)\,,$$

and finally,

$$\phi(x,t) = \sum_{n=1}^{\infty}\frac{w_n}{a^2\lambda_n^2}\left(1 - e^{-a^2\lambda_n^2 t}\right)\sin(\lambda_n x) + t - \frac{cxt}{1+cL}\,,$$

where w_n and λ_n are as previously defined.

2. Show that the equation

$$u_t = u_{xx} + Q(x)\,, \quad 0 \le x \le L\,, \quad t > 0\,,$$

with the boundary conditions $u_x(0) = u_x(L) = 0$ has no equilibrium solution unless $\int_0^L Q(x)dx = 0$. In other words show an insulated bar, to which energy is being added (or subtracted), can not obtain a thermal equilibrium.

This question if actually quite strait forward. For an equilibrium solution, we set $u_t = 0$. So we have $u_{xx} + Q(x) = 0$. We integrate this from 0 to L,

$$\begin{aligned} \int_0^L u_{xx}\, dx &= -\int_0^L Q(x)\, dx\,, \\ u_x\big|_0^L &= -\int_0^L Q(x)\, dx\,, \\ 0 &= -\int_0^L Q(x)\, dx\,. \end{aligned}$$

So for an equilibrium (or steady-state) to exist, we must require $\int_0^L Q(x)\, dx = 0$

3. Find the solution, $u(x, y, t)$, to the following PDE:

$$u_t = k(u_{xx} + u_{yy})\,, \quad 0 \le x \le 1\,, \quad 0 \le y \le 1\,, t > 0\,,$$

with the boundary conditions $u_x(0, y, t) = u_x(1, y, t) = 0$, $u(x, 0, t) = 0$, $u(x, 1, t) = 1$ and initial conditions $u(x, y, 0) = \phi(x, y)$. **Note: you may leave your answer in the form of an infinite series, but be sure to define all the terms of the series in terms of the given functions**.
The first thing we must do is to make the boundary conditions homogeneous. We thus let $u(x, y, t) = v(x, y, t) + z(x, y, t)$ We want to find a function z such that v will satisfy a similar PDE with homogeneous boundary conditions. As in question 1, this will result in the following conditions for z:

$$\begin{aligned} z_x(0, y, t) &= z_x(1, y, t) = 0\,, \\ z(x, 0, t) &= 0\,, \\ z(x, 1, t) &= 1\,. \end{aligned}$$

After looking at these conditions, I will try $z(x, y, t) = a + by$. With this choice we find that $z(x, y, t) = y$.
Now v will satisfy the following PDE:

$$\begin{aligned} v_t = k(v_{xx} + v_{yy})\,, \quad 0 \le x \le 1\,, &\quad 0 \le y \le 1\,, t > 0\,, \\ v_x(0, y, t) &= v_x(1, y, t) = 0\,, \\ v(x, 0, t) &= v(x, 1, t) = 0\,, \\ v(x, y, 0) &= \phi(x, y) - y\,. \end{aligned}$$

Since I have already solved similar problems in class, I will just write out the form of the solution.

$$v(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-(n^2+m^2)\pi^2 kt} \cos(m\pi x) \sin(n\pi y)\,.$$

Note that we must include the terms for which $m = 0$ as these are still non-trivial solutions to the eigenvalue problem. Now we use the initial conditions to find,

$$C_{mn} = 4 \int_0^1 \int_0^1 (\phi(x, y) - y) \sin(n\pi y) \cos(m\pi x)\, dx\, dy\,.$$

Finally we have,

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn} e^{-(n^2+m^2)\pi^2 kt} \sin(n\pi y)\cos(m\pi x) + y$$

4. Use the Rayleigh quotient to obtain an upper bound for the lowest eigenvalue of
   (a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(1) = 0$.
   First we must come up with a test function which satisfies the boundary conditions. To keep things simple we can use $u = x^2 - 1$. The Rayleigh quotient is given by,

$$\begin{aligned} RQ(u) &= -\frac{\int_0^1 uLu\, dx}{\int_0^1 u^2\, dx} \\ &= \frac{-u\frac{du}{dx}|_0^1 + \int_0^1 (\frac{du}{dx})^2 + x^2u^2\, dx}{\int_0^1 u^2\, dx} \\ &= \frac{37}{14}. \end{aligned}$$

   So we have $\lambda_1 \leq \frac{37}{14}$.
   (b) $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) + \phi(1) = 0$.
   For this problem I will let $u = ax^2 + bx + c$. $\frac{d\phi}{dx}(0) = 0$ tells us that $b = 0$. $\frac{d\phi}{dx}(1) + \phi(1) = 0$ tells us that $2a + c + a + c = 0$. If we let $c = 3$ that $a = 1$. So we will use $u = x^2 - 3$. Now

$$\begin{aligned} RQ(u) &= -\frac{\int_0^1 uLu\, dx}{\int_0^1 u^2\, dx} \\ &= \frac{-u\frac{du}{dx}|_0^1 + \int_0^1 (\frac{du}{dx})^2 + xu^2\, dx}{\int_0^1 u^2\, dx} \\ &= \frac{85}{72}. \end{aligned}$$

5. Consider the eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0\,,$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).
We use the Rayleigh quotient. We pick a function u which satisfies the boundary conditions and we have,

$$\begin{aligned} \lambda_1 &\leq RQ(u)\,, \\ &= -\frac{\int_0^1 uLu\, dx}{\int_0^1 u^2\, dx} \\ &= \frac{-u\frac{du}{dx}|_0^1 + \int_0^1 (\frac{du}{dx})^2 + x^2u^2\, dx}{\int_0^1 u^2\, dx} \\ &= \frac{\int_0^1 (\frac{du}{dx})^2 + x^2u^2\, dx}{\int_0^1 u^2\, dx}. \end{aligned}$$

First,we note that $RQ(u) \geq 0$ since each all of the terms in the integrand is greater than or equal to 0. Now all we have to show is that the integrand cannot be identically 0.
If the integrand is identically zero then,

$$(\frac{du}{dx})^2 = -x^2u^2$$

So $u = Ce^{\pm i\frac{x^2}{2}}$. The only way to satisfy the boundary conditions is $C = 0$ so the Rayleigh quotient must be strictly greater than 0 and we are done.

6. A rod occupies the interval $1 < x < 2$. The thermal conductivity depends on x in such a way that the temperature $\phi(x,t)$ satisfies the equation,

$$\phi_t = A^2(x^2\phi_x)_x$$

where A is a constant. For $\phi(1,t) = 0 = \phi(2,t)$ for $t > 0$ and $\phi(x,0) = f(x)$ for $1 < x < 2$, show that the appropriate eigenfunctions β_n are,

$$\beta_n(x) = \frac{1}{\sqrt{x}}\sin\left(\frac{\pi n\ln x}{\ln(2)}\right)$$

and work out the separation of variables solution of this problem.
First we find guess that $\phi(x,t) = X(x)T(t)$ and plug in to get the usual,

$$\frac{T'}{A^2T} = x^2X'' + 2xX'X = -\lambda\,.$$

First lets solve the eigenfunction problem we get from the X equation.

$$x^2X'' + 2xX + \lambda X = 0\,,$$

is an Euler equation. We make the guess $X = x^\alpha$, and plug it in. We find that $\alpha^2 + \alpha + \lambda$, so $\alpha = \frac{-1\pm\sqrt{1-4\lambda}}{2}$. We must consider three cases. First if $1 - 4\lambda > 0$ then,

$$X = Ax^{\alpha_1} + Bx^{\alpha_2}\,,$$

where $\alpha_{1,2}$ are, $\frac{-1-\sqrt{1-4\lambda}}{2}$ and $\frac{-1+\sqrt{1-4\lambda}}{2}$. If we apply the boundary condition at $x = 1$, we get $A = -B$. Then we apply the boundary condition at 2 and we find $A(2^{\alpha_1} - 2^{\alpha_2}) = 0$. Since $\alpha_1 \neq \alpha_2$, we have $A = 0$, so we may ignore this case.
The next case is $1 - 4\lambda = 0$, or $\alpha_1 = \alpha_2 = \alpha$. For this case, the solution to the equation is,

$$X = C_1x^\alpha + C_2x^\alpha\ln(x)\,.$$

Now $X(1) = C_1 = 0$ and $X(2) = C_22^\alpha\ln(2) = 0$. So $C_2 = 0$ as well and we can disregard this case. The only remain case is for $\alpha_{1,2}$ to be complex conjugates, $\alpha_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{4\lambda-1}}{2}$. We can take the plus case for now. So we can write one solution of the problem as,

$$\begin{aligned} X(x) &= x^{-\frac{1}{2}+i\frac{\sqrt{4\lambda-1}}{2}}\,, \\ &= e^{\ln(x^{-\frac{1}{2}+i\frac{\sqrt{4\lambda-1}}{2}})}\,, \\ &= e^{(-\frac{1}{2}+i\frac{\sqrt{4\lambda-1}}{2})\ln(x)}\,, \\ &= \frac{1}{\sqrt{x}}e^{i\frac{\sqrt{4\lambda-1}}{2}\ln(x)}\,. \end{aligned}$$

Using Euler's identity, we can write two independent solutions as, $X_1 = \frac{1}{\sqrt{x}}\cos(\frac{\sqrt{4\lambda-1}}{2}\ln(x))$ and $X_2 = \frac{1}{\sqrt{x}}\sin(\frac{\sqrt{4\lambda-1}}{2}\ln(x))$. The general solutions is just $X(x) = AX_1(x) + BX_2(x)$. We apply the boundary condition at $x = 1$ and find $A = 0$, now we apply the boundary condition at $x = 2$ and get,

$$\frac{1}{\sqrt{2}}\sin(\frac{\sqrt{4\lambda-1}}{2}\ln(2)) = 0\,.$$

So $\frac{\sqrt{4\lambda-1}}{2}\ln(2) = n\pi$, $n = 1, \ldots$. This gives us,

$$\lambda_n = \frac{n^2\pi^2}{\ln^2(2)} + \frac{1}{4}\,,$$

with eigenfunction, $\beta_n(x)$. To get the separation of variables solution, we solve the time equation $T_n' + \lambda_n A^2 T_n = 0$, which gives us $T_n(x) = T_n(0)e^{-\lambda_n A^2 t}$ with λ_n defined above. We will then guess our solution to be in the form,

$$\phi(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n A^2 t}\beta_n(x)\,.$$

Now we use the initial conditions to solve for the C_n's. Before we do that, we note that the β_n's haven't been normalized. So,

$$C_n = \frac{1}{\|\beta_n\|_2^2}\int_1^2 f(x)\beta_n(x)\,dx$$

In this case it is very easy to find $\|\beta_n\|_2^2$. It is

$$\begin{aligned}\|\beta_n\|_2^2 &= \int_1^2 \frac{1}{x}\sin^2\left(\frac{\pi n \ln x}{\ln(2)}\right)\,dx \\ &= \frac{\ln(2)}{2}\end{aligned}$$

So,

$$C_n = \frac{2}{\ln(2)}\int_1^2 f(x)\beta_n(x)\,dx$$