**Math 4220/5220 -Introduction to PDE's**

Homework #3 Due March 13

1. Find the 1-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be stated as follows:
   (a) It solves $G''(x) = 0$ for $x \neq x_0$.
   (b) $G(0) = G(l) = 0$.
   (c) $G(x)$ is continuous at $x_0$ and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at $x_0$.
2. (a) Find the Green's function for the half-plane $\{y > 0\}$.
   (b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.
   (c) Calculate the solution with $u(x, 0) = 1$.
3. (a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by $f$.
   (b) Show that $\frac{\partial u}{\partial r} \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.
   (c) Suppose that $v(x, y)$ is any function in $\{y > 0\}$ such that $\frac{\partial v}{\partial r} \equiv 0$. Show that $v$ is a function of the quotient $x/y$.
   (d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$.
   (e) Show that your answer to (c) and (d) agrees with the general formula in the previous question.
4. (a) Use the previous question to find a harmonic function in the half-plane $\{y > 0\}$ with the boundary data $h(x) = 1$ for $x > 0$, $h(x) = 0$ for $x < 0$.
   (b) Do the same as part (a) for the boundary data $h(x) = 1$ for $x > a$, $h(x) = 0$ for $x < a$.
   (c) Use part (b) to solve the same problem with boundary data $h(x)$, where $h(x)$ is any step function. That is
   $$h(x) = c_j \text{ for } a_{j-1} < x < a_j \text{ for } 1 \leq j \leq n$$
   where $-\infty = a_0 < a_1 < a_2 \cdots < a_{n-1} < a_n = \infty$ and the $c_j$ are constants.
5. Find the Green's function for the half-ball $\{x^2 + y^2 + z^2 < a^2, z > 0\}$. Hint: The easiest method is to use reflections with the solution for the whole ball.
6. Do the same for the eighth of a ball $D = \{x^2 + y^2 + z^2 < a^2, x > 0, y > 0, z > 0\}$.

For students enrolled in Math 5220 all questions are of equal value and required.

For students enrolled in Math 4220, question 4 is a bonus question worth up to an additional 5 %.