Math 4220/5220 -Introduction to PDE's
Homework #3 Solutions

1. Find the 1-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be stated as follows:
   (a) It solves $G''(x) = 0$ for $x \neq x_0$.
   (b) $G(0) = G(l) = 0$.
   (c) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

   The first property gives us that

$$G(x) = \begin{cases} a_0 x + b_0 & 0 < x < x_0 \\ a_1 x + b_1 & x_0 < x < l \end{cases}$$

   Now we apply $G(0) = G(1) = 0$ and the fact that G is continuous at x_0 to get

$$G(x) = \begin{cases} ax & 0 < x < x_0 \\ \frac{ax_0}{x_0 - l}(x - l) & x_0 < x < 1 \end{cases}$$

   We just need to use the fact that $R(x) = G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0 to find a. The fact that this function is harmonic implies that its first derivative is continuous at x_0. A simple calculation gives:

$$\lim_{x \to x_0 -} R'(x) = a - \frac{1}{2}$$

   and

$$\lim_{x \to x_0 +} R'(x) = \frac{ax_0}{x_0 - l} + \frac{1}{2}$$

   Equating these limits results in

$$a = 1 - \frac{x_0}{l}$$

   So

$$G(x) = \begin{cases} \left(1 - \frac{x_0}{l}\right) x & 0 < x < x_0 \\ -\frac{x_0}{l}(x - l) & x_0 < x < l \end{cases}$$

2. (a) Find the Green's function for the half-plane $\{y > 0\}$.
   Given $\mathbf{x}_0 = (x_0, y_0)$ in the upper half-plane, we define its reflection $\mathbf{x}_0^* = (x_0, -y_0)$. Let

$$G(\mathbf{x}; \mathbf{x}_0) = f(\mathbf{x} - \mathbf{x}_0) - f(\mathbf{x} - \mathbf{x}_0^*)$$

   where $f(\mathbf{y}) = \frac{1}{2\pi} \ln(|\mathbf{y}|)$. It is clear that G is harmonic in the upper half-plane with $\mathbf{x}_0$ removed and we have the right type of singularity at $\mathbf{x}_0$. We just need to check the value on the boundary. Since when $\mathbf{x}$ is on the x-axis, $|\mathbf{x} - \mathbf{x}_0| = |\mathbf{x} - \mathbf{x}_0^*|$, G will be 0 on the boundary.

   (b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.
   On the boundary of D, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial y}$, so on the boundary of D, we have

$$\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}_0) = \frac{y_0}{\pi((x - x_0)^2 + y_0^2)}$$

The solution on the half-plane is then given by

$$u(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{h(x)}{(x - x_0)^2 + y_0^2}\, dx\,.$$

(c) Calculate the solution with $u(x, 0) = 1$.

$$u(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x - x_0)^2 + y_0^2}\, dx = \left[\frac{1}{\pi} \arctan(x/y_0)\right]_{-\infty}^{\infty} = 1\,.$$

So $u(x, y) \equiv 1$.

3. (a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.
We differentiate to get:

$$\begin{aligned} u_x &= \frac{1}{y} f'(x/y) \\ u_{xx} &= \frac{1}{y^2} f''(x/y) \\ u_y &= -\frac{x}{y^2} f'(x/y) \\ u_{yy} &= \frac{x^2}{y^4} f''(x/y) + \frac{2x}{y^3} f'(x/y) \end{aligned}$$

So,

$$u_{xx} + u_{yy} = \left(\frac{1}{y^2} + \frac{x^2}{y^4}\right) f''(x/y) + \frac{2x}{y^3} f'(x/y) = 0$$

We factor out a $\frac{1}{y^2}$ and let $s = \frac{x}{y}$ to get the following ODE for f:

$$\begin{aligned} (1 + s^2) f''(s) + 2s f'(s) &= 0 \\ ((1 + s^2) f'(s))' &= 0 \\ f'(s) &= \frac{C_1}{1 + s^2} \\ f(s) &= C_1 \arctan(s) + C_2 \end{aligned}$$

(b) Show that $\frac{\partial u}{\partial r} \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{y} f'\left(\frac{x}{y}\right)\frac{x}{r} - \frac{x}{y^2} f'\left(\frac{x}{y}\right)\frac{y}{r} = 0\,.$$

(c) Suppose that $v(x, y)$ is any function in $\{y > 0\}$ such that $\frac{\partial v}{\partial r} \equiv 0$. Show that v is a function of the quotient x/y.
If $\frac{\partial v}{\partial r} \equiv 0$, the v is a function of just θ, but $\theta = \arctan(y/x)$ is a function of just $\frac{x}{y}$, so v must also be a function of just $\frac{x}{y}$.

(d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$.
From the solution in part (a), we have

$$\lim_{y \to 0} u(x, y) = \lim_{y \to 0} C_1 \arctan(y/x) + C_2 = \begin{cases} \frac{\pi}{2} C_1 + C_2 & x > 0 \\ -\frac{\pi}{2} C_1 + C_2 & x < 0 \end{cases}$$

(e) Show that your answer to (c) and (d) agrees with the general formula in the previous question.
We plug in the boundary data,

$$h(x) = \begin{cases} \frac{\pi}{2}C_1 + C_2 & x > 0 \\ -\frac{\pi}{2}C_1 + C_2 & x < 0 \end{cases}$$

into the solution found in (2b) to get

$$\begin{aligned} u(x_0, y_0) &= \frac{y_0}{\pi}\int_{-\infty}^{0} \frac{-\frac{\pi}{2}C_1 + C_2}{(x-x_0)^2 + y_0^2}\,dx + \frac{y_0}{\pi}\int_{-\infty}^{0} \frac{\frac{\pi}{2}C_1 + C_2}{(x-x_0)^2 + y_0^2}\,dx \\ &= \frac{-\frac{\pi}{2}C_1 + C_2}{\pi}\arctan\left(\frac{x-x_0}{y_0}\right)\Bigg|_{-\infty}^{0} + \frac{\frac{\pi}{2}C_1 + C_2}{\pi}\arctan\left(\frac{x-x_0}{y_0}\right)\Bigg|_{0}^{\infty} \\ &= C_1\arctan(x_0/y_0) + C_2 \end{aligned}$$

4. (a) Use the previous question to find a harmonic function in the half-plane $\{y > 0\}$ with the boundary data $h(x) = 1$ for $x > 0$, $h(x) = 0$ for $x < 0$.
We Just set $C_1 = \frac{1}{\pi}$ and $C_2 = \frac{1}{2}$ and we have it.

(b) Do the same as part (a) for the boundary data $h(x) = 1$ for $x > a$, $h(x) = 0$ for $x < a$.
We just need to translate the solution by a,

$$u(x, y) = \frac{1}{\pi}\arctan(y/(x-a)) + \frac{1}{2}$$

(c) Use part (b) to solve the same problem with boundary data $h(x)$, where $h(x)$ is any step function. That is

$$h(x) = c_j \text{ for } a_{j-1} < x < a_j \text{ for } 1 \le j \le n$$

where $-\infty = a_0 < a_1 < a_2 \cdots < a_{n-1} < a_n = \infty$ and the c_j are constants.
As we cross each point a_i on the x-axis, we want the solution to increase by $C_i - C_{i+1}$.
We can write the solution as,

$$u(x, y) = \sum_{i=1}^{n-1} \frac{C_{i+1} - C_i}{\pi}\arctan(y/(x-a_i)) + \frac{C_n + C_1}{2}.$$

Note that if $x > a_{n-1}$ then we get

$$\frac{C_2 - C_1}{2} + \frac{C_3 - C_2}{2} + \cdots + \frac{C_n - C_{n-1}}{2} + \frac{C_n + C_1}{2}$$

Which telescopes down to C_n. If we are between a_{n-2} and a_{n-1} then we would have

$$\frac{C_2 - C_1}{2} + \frac{C_3 - C_2}{2} + \cdots + \frac{C_{n-1} - C_{n-2}}{2} - \frac{C_n - C_{n-1}}{2} + \frac{C_n + C_1}{2}$$

which telescopes down to C_{n-1} as required. It should be clear that we get the correct values for the rest of the x-axisgr.

5. Find the Green's function for the half-ball $\{x^2 + y^2 + z^2 < a^2, z > 0\}$. Hint: The easiest method is to use reflections with the solution for the whole ball.

For $\mathbf{x}_0 = (x_0, y_0, z_0)$ in the half ball D, define $\mathbf{x}_0^* = (x_0, y_0, -z_0)$. Let G_B denote the solution in the whole ball, then

$$G(\mathbf{x}; \mathbf{x}_0) = G_B(\mathbf{x}; \mathbf{x}_0) - G_B(\mathbf{x}; \mathbf{x}_0^*)$$
$$= \frac{1}{4\pi|\mathbf{x} - \mathbf{x_0}|} - \frac{1}{4\pi|\mathbf{x} - \bar{\mathbf{x_0}}|} - \frac{1}{4\pi|\mathbf{x} - \mathbf{x_0^*}|} - \frac{1}{4\pi|\mathbf{x} - \bar{\mathbf{x_0^*}}|}$$

In the above expression, the over bar means the point reflected through the sphere as discussed in class.

6. Do the same for the eighth of a ball $D = \{x^2 + y^2 + z^2 < a^2, x > 0, y > 0, z > 0\}$.
Each time we eliminate a piece of the sphere, we must double the number of points. We start with 2 points. To get to a half-sphere, we must use 4 points. To get to a quarter-sphere, we will need 8 points. Finally for the eighth-sphere, we will need 16 points. Each new set of points will come form reflecting the old set across the axis.

For students enrolled in Math 5220 all questions are of equal value and required.

For students enrolled in Math 4220, question 4 is a bonus question worth up to an additional 5 %.