Manipulating polynomials

Let $a(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ **and** $b(x) = b_0 + b_1x + b_2x^2 + b^3x^3$**.**

Product:

$$a(x)b(x) = \sum_n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.$$

The coefficient of x^n **in the product is**

$$\sum_{k=0}^{n} a_k b_{n-k} = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots + a_{n-1}b_1 + a_nb_0.$$

The sequence formed by the coefficients of the product is called a *convolution* **of the sequences** a_n **and** b_n**.**

Multiplication by x:

$$xa(x) = a_0x + a_1x^2 + a_2x^3 + \ldots = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Similarly,

$$x^t a(x) = a_0x^t + a_1x^{t+1} + a_2x^{t+2} + \ldots = \sum_{n=t}^{\infty} a_{n-t}x^n.$$

Note that the coefficients of $x^0, x^1, \ldots x^{t-1}$ are zero in this product.

Division by x:

$$\frac{a(x)}{x} = a_0\frac{1}{x} + a_1 + a_2x + a_3x^2 + \ldots = \sum_{n=0}^{\infty} a_{n+1}x^n \; - \frac{a_0}{x}$$

And thus also:

$$\frac{a(x)}{x^t} = a_0\frac{1}{x^t} + \ldots + a_t + a_{t+1}x + a_{t+2}x^2 + \ldots = \sum_{n=0}^{\infty} a_{n+t}x^n - \frac{a_0}{x^t} - \ldots - \frac{a_{t-1}}{x}$$

So multiplication and division shift the sequence to the right or the left.

Substitution

Example 1:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \dots$$

Substitute $z = -2x^2$:

$$\frac{1}{1+2x^2} = 1 - 2x^2 + 4x^4 - 8x^6 + 16x^8 - \dots$$

Example 2:

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

Substitute $z = 3x^2$:

$$(1+3x^2)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k x^{2k}$$

Take derivatives:

$$a'(x) = \frac{d}{dx}a(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

$$a''(x) = \frac{d^2}{dx^2}a(x) = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \dots$$

In general (t-th derivative):

$$a^{(t)}(x) = \sum_{n=t}^{\infty} n(n-1)\cdots(n-t+1)a_n x^{n-t}.$$

Example:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Bonus question:

Use generating functions to prove the following relation about Fibonacci numbers:

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{2nF_{n+1} - (n+1)F_n}{5}$$

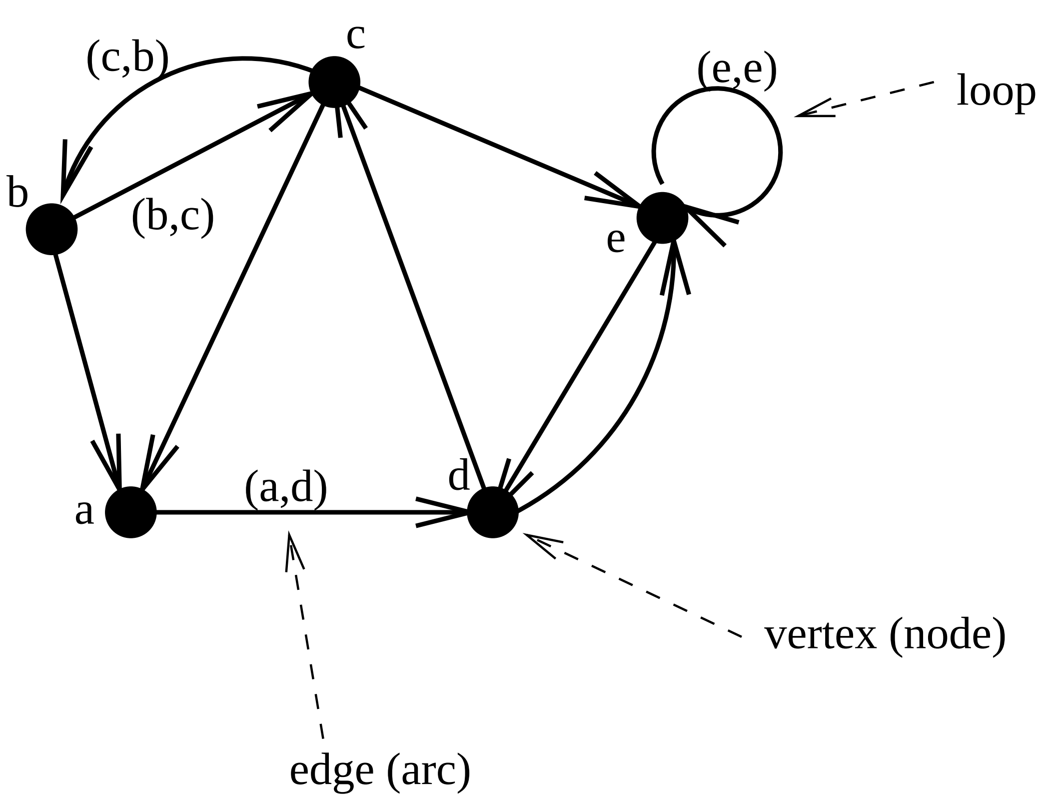
Hints: left-hand side is the coefficient of a product. Also, use the fact that

$$f(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{x}{r_1}} - \frac{1}{1 - \frac{x}{r_2}} \right),$$

where r_1 **and** r_2 **are the roots of** $x^2 - x - 1$. **You can verify that** $(1/r_1) + (1/r_2) = 1$. **See also the notes of class 4.1.**

Graph Theory

Example: directed graph.



bcb **is a (directed) cycle of length 2.**
ceed **is a trail of length 3, but not a path.**
cadc **is a cycle of length 3, and hence also a circuit and a closed walk.**
cab **is not a path, because there is no edge** (a, b)**.**

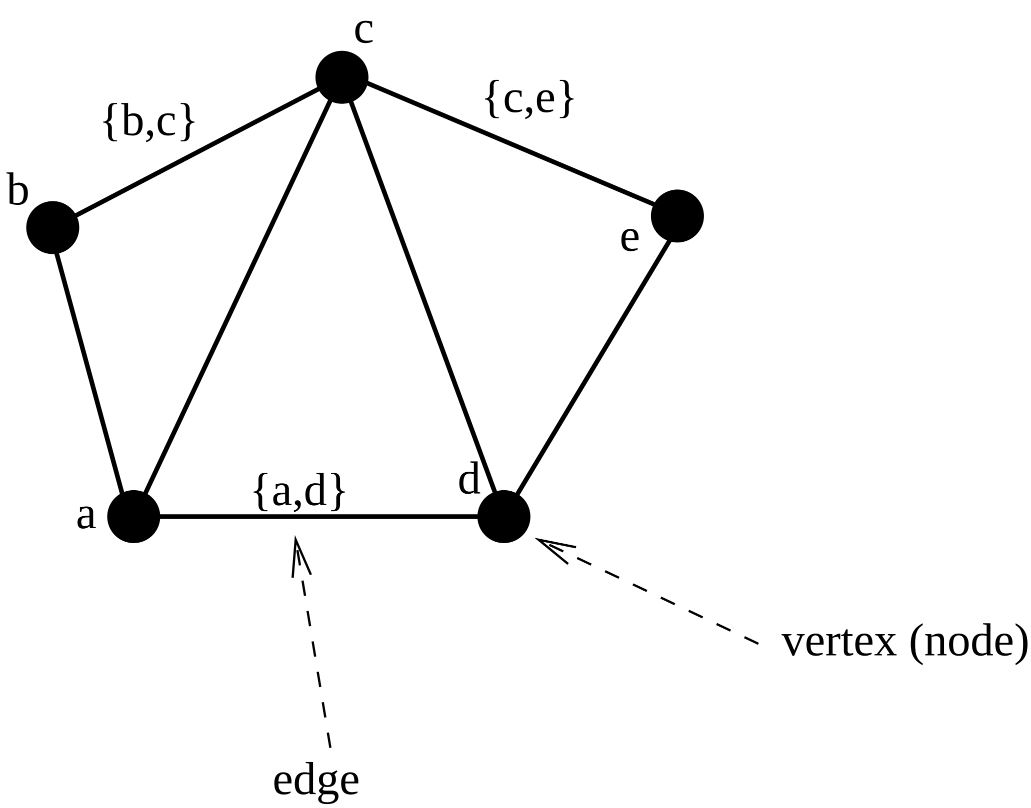
The length of a walk is the number of edges in the walk.

A *closed walk* **is a walk where the first and last vertex are equal.**

A *trail* **is a walk without repeated edges. A closed trail is called a** *circuit***.**

A *path* **is a walk without repeated nodes. A closed path is called a** *cycle***. (Note that in a cycle, the only repeated node is the begin/endpoint.) The definitions of closed walk, trail, circuit, path and cycle are analogous to those for directed graphs.**

Example: undirected graph.



cedc **is a cycle of length 3.**
cedca **is a trail and a walk, but not a path.**
ceced **is a walk, but not a trail or a path.**
cedcabc **is a circuit, but not a cycle.**

An undirected graph G is *connected* if there is a path between any two vertices of G.

Any graph that is not connected can be decomposed into a number of disconnected components.

$\kappa(g)$ **denotes the number of components of G.**