MATH 3330 — Applied Graph Theory

Assignment 3

Due Tuesday, January 30, 2006 (before class)

1. Give the number of 4-cycles in the hypercubes Q_2 and Q_3. Using the iterative way of constructing Q_n, derive a formula for the number of 4-cycles in Q_4. Finally, give a general formula for the number of 4-cycles in Q_n, for all $n \geq 2$.

2. Suppose that $G = (V, E)$ is the intersection graph of a collection of sets S_v, which are such that the union of all the S_v contain only four elements. (So $|\bigcup_{v\in V} S_v| = 4$.)

   (a) Can G contain an induced 4-cycle? If no, explain (in detail) why not. If yes, give an example.

   (b) Can G contain an induced 5-cycle? If no, explain (in detail) why not. If yes, give an example.

3. A telephone company has to decide which of its lines to use as backbones, to route long distance traffic. It wants to connect n relay stations, which already have lines connecting them. For each existing line ℓ_{ij} connecting stations i and j, the probability of failure is known. The probability of failure of the lines are considered independent, so the probability that, at any time, two lines ℓ_{ij} and ℓ_{km} are both not failing equals the product $(1 - p_{ij})(1 - p_{km})$. The company wants to choose a backbone network that will form a *spanning tree* connecting all n stations, but so that the probability of the network failing is minimized (the network fails if *any* of its lines fails, because then the backbone is disconnected.)

   (a) An engineer with the company explains that this is problem is equivalent to the minimum spanning tree problem, if the weight of each edge ℓ_{ij} is taken to be $-\log{(1 - p_{ij})}i$. Explain why the engineer is correct.

   (b) The engineer proposes the following algorithm: Order all lines in order of increasing failure probability (so the first line will be the one with lowest failure probability, etc.). Then, choose lines in order. If a line does not form a cycle with lines already chosen, add it to the spanning tree. Otherwise, ignore it. Give a reasoning that this algorithm is equivalent to Kruskal's algorithm applied to the weights described in (a).

(c) Demonstrate the algorithm on a network of 5 stations labelled 1,2,3,4,5. The failure probabilities are given in the table below.

	2	3	4	5
1	0.01	0.01	0.02	0.005
2		0.015	0.017	0.005
3			0.017	0.02
4				0.007

4. Give a detailed argument why Kruskal's algorithm cannot be used to find a minimum directed spanning tree in a strongly connected graph.

5. Do problems 3.1.6–3.1.8 of the text.

6. Do problem 1.4.36 of the text.