MATH 3330 — Applied Graph Theory
Assignment 1 — Solutions

(1.1.19) Determine, with the methods shown in class, whether each of the following sequences is graphic. If it is, draw a graph that realizes the sequence.

a. (7,6,6,5,4,3,2,1) b.(5,5,5,4,2,1,1,1)
c. (7,7,6,5,4,4,3,2) d.(5,5,4,4,2,2,1,1)

Use the reduction method shown in class, and in the text, page 9. The sequences from a. and b. are not graphic, the others are. Note that you were required to draw the whole graph realizing the sequence (in one picture). 2 points

(1.1.26,27) A pair of sequences $< a_1, \ldots, a_n >$ and $< b_1, \ldots, b_n >$ is *digraphic* if there exists a simple digraph (digraph with no multi-edges or self-loops) with vertex-set $\{v_1, \ldots, v_n\}$ so that $outdegree(v_i) = a_i$ and $indegree(v_i = b_i)$ for $i = 1, \ldots, n$.

(a) *Note first that the digraph must be simple, so no loops or multiple edges are allowed (two arcs in opposite directions between a pair of vertices is allowed). Many students gave a method which would lead to a digraph which is not necessarily simple. A reduction method to determine whether the pair of sequences is digraphic can be developed along the same lines as the method used in the first question to see whether a sequence is graphic. Namely: Given sequences* $< a_1, a_2, \ldots, a_n >$ *and* $< b_1, b_2, \ldots, b_n >$*, reduce to smaller sequences as follows: remove* a_1 *and* b_1x *from the sequences. Reduce the* a_1 *elements* b_i $(i \geq 2)$ *by one, and the* b_1 *highest elements of* a_i $(i \geq 2)$ *by one. Recursively determine if these sequences are digraphic. If the new pair is digraphic, we can extend the graph by adding a new vertex* v_1*, and make out-edges from* v_1 *to the* a_1 *vertices whose vertices whose in-degree is* $b_i - 1$*, and in-edges from the* b_1 *vertices whose out-degree is* $a_i - 1$*. Clearly, this new digraph has the required in- and out-degrees.*

If the reduced pair of sequences is not digraph, the original pair is not digraph either. The argument is similar to the one for the original method. The key statement to prove is: If there exists a digraph with the required in- and out-degrees, then there also *exists a digraph with the required in- and out-degrees, AND so that v_1 has as out-neighbours the a_1 vertices of highest in-degree, and as in-neighbours the b_1 vertices with highest out-degree. The argument is that any graph can be transformed using 2-switches so that the extra property holds and the out- and in-degrees remain the same. 2 points.*

(b) Use your method to determine whether the pair of sequences $< 3, 1, 1, 0 >$ and $< 1, 1, 1, 2 >$ is digraphic. Show your work.

Reduce to the pair $< 0, 1, 0 >$ and $< 0, 0, 1 >$. Obviously, this pair can be realized by forming a graph with vertices v_2, v_3, v_4 with one edge from v_3 to v_4. Now add a vertex v_1 with out-edges to v_2, v_3, v_4, and in-edges from v_2. 1 point

(1.2.2) What is the maximum possible number of edges in a simple bipartite graph on m vertices? (Explain your answer)

The maximum number of edges in a bipartite graph is achieved by a complete bipartite graph. The number of edges in a complete bipartite graph $K_{a,b}$ equals ab. If there are a total of m vertices, then we need to maximize ab, subject to the condition that $a+b=m$. It is a fairly easy calculus problem that the maximum is achieved when $a = b = m/2$, which gives $m^2/4$ edges. If m is even, this works; if m is even, the best we can do is $a = (m-1)/2$, $b = (m+1)/2$, which gives $(m^2-1)/4$ edges. 2 points.

(1.2.28) Show that every simple graph is an intersection graph by describing (in general) how to construct a family of sets which it represents.

Let $G = (V, E)$ be any simple graph. Then for each vertex v, form the set S_v containing all edges of which v is an endpoint. Then G is the intersection graph of the S_v: if vertices u and v are adjacent, then both sets S_u and S_v contain the edge $\{u, v\}$, and thus $S_u \cap S_v \neq \emptyset$. Conversely, if $S_u \cap S_v \neq \emptyset$, then there must be an edge $e \in S_u \cap S_v$. Since $e \in S_u$, u must be an endpoint of e. Since $e \in S_v$, v must also be an endpoint of e. So e must be the edge $\{u, v\}$, and u and v must be adjacent. So for any pair of vertices u and v, u and v are adjacent in G if and only if $S_u \cap S_v \neq \emptyset$. Bonus 2 points

(1.4.21–24) Determine the diameter, radius, and central vertices of the following graphs: *3 points*

(a) Path graph P_n. *Diameter* n*, radius* $\lceil (n-1)/2 \rceil$*, central vertices the* $(n+1)/2$*-th vertex if* n *is odd, the* $n/2$*-th and* $n/2+1$*-th vertex if* n *is even.*

(b) Cycle graph C_n. *Diameter equals radius:* $\lfloor n/2 \rfloor$*. Central vertices: all.*

(c) Complete graph K_n. *Diameter equals radius: 1. Central vertices: all.*

(d) Complete bipartite graph $K_{n,m}$. *If* n *and* m *both greater than 1, then diameter equals radius equals 2, all vertices are central. If* $n = 1$ *and* $m > 1$ *or vice versa, then diameter equals 2, radius equals 1, central vertex is the vertex on the bipartite side of size 1. If* $n = m = 1$*, then the graph is isomorphic to* K_2*, see* d*.*

(e) Petersen graph. *Diameter equals radius equals 2. Central vertices: all.*