MATH 3330 — Applied Graph Theory
Assignment 10 – Solutions

1. (9.1.9 and 9.1.16) Find a proper colouring of the graph of 9.1.8 which uses the miminum number of colours. Give an argument why this is indeed the minimum number of colours. Then apply the largest-degree-first heuristic (greedy colouring where the vertices are ordered so that vertices with larger degree come first).

   A colouring with four colours can be found: colour around the cycle, top vertex first: 123412343, and center vertex 2.

   The graph has triangles, so it needs at least three colours. Consider the subgraph induced by the top vertex and its five neighbours. It is easy to verify, trying all possibilities, that this graph cannot be coloured with four colours. There are several ways to do the largest-degree-first heuristic. They all colour optimally. One example (in case of equal degree, I coloured topmost vertices first) gives a colouring of 132321212 clockwise around the outer cycle, and 4 in the middle.

2. Consider the collection of intervals given by the table below. (a) Draw the interval graph corresponding to this collection. (b) Find the clique number of this graph. (c) Show how to apply greedy colouring to obtain an *optimal* colouring of this graph. Explain your work.

Interval	start time	end time
a	10:01	10:35
b	9:05	11:35
c	10:05	10:15
d	9:55	10:45
e	8:30	9:35
f	10:55	11:15
g	11:00	12:05
h	8:05	10:10

   Draw graph: make a vertex for each interval, and add edges between vertices whose intervals overlap. The clique number is 6, the clique is formed by vertices a, b, c, d, e, h. *To obtain an optimal colouring using the greedy approach, order the vertices according to the start time:* $h - e - b - d - a - c - f - g$, *and colour according to this order. Colours assigned, in this order, are* $1 - 2 - 3 - 4 - 5 - 6 - 1 - 2$.

3. For which values of n is the wheel W_n perfect? Explain your answer in detail.

   The wheel W_n consists of an n-cycle plus one center vertex attached to all vertices of the cycle.

   For $n = 3$, this results in 4 vertices that are all connected. So W_3 is isomorphic to K_4, which is perfect.

   For $n > 3$ and odd, W_n is not perfect. Namely, its clique number is 3, but you will need 4 colours to colour it: 3 for the odd cycle, and one additional colour for the center vertex.

   For $n > 3$ and even, W_n is perfect. Note that W_n has clique number 3, and can be coloured with 3 colours. Consider any subgraph H of W_n. If H contains a triangle, then the clique number of H equals 3, and H can be coloured with 3 colours, so $\omega(H) = \chi(H)$. Suppose then that $\omega(H) = 2$. So H does not contain a triangle. Note that all odd cycles in W_n contain a triangle. Therefore, if H does not contain a triangle it does not contain an odd cycle. Therefore, H is bipartite, and thus can be coloured with 2 colours. Finally, if $\omega(H) = 1$ then H is a collection of isolated vertices, so can be coloured with 1 colour.

4. (6.3.8) Draw an 8-vertex simple hamiltonian graph with more than 8 edges that is not eulerian, or give an argument why it does not exist.

   Exists: for example, draw an 8-cycle with one chord, then two vertices will have odd degree, so graph is not eulerian.

5. (6.4.4) Apply each of the algorithms 6.4.1 (nearest neighbour), 6.4.2 (double the tree) and 6.4.3 (tree and matching) to the given TSP problem. Show your work.

   Nearest neighbour gives the tour: $acedbfa$, cost 36.

   MST consists of edges de, ef, ac, bd, ec. A tour, with duplication, of going around the tree is: $acedbdfeca$. Removing duplicates, we get: $acedbfa$, of cost 36 (same tour as nearest neighbour).

   Tree plus matching: All vertices in the MST except c and d have odd degree. The minimum cost perfect matching between vertices a, b, e, f is ab, ef. Adding these edges to the MST, we obtain an eulerian graph. making an eulerian tour, we get $abdefeca$. Removing duplicates, we get the tour $abdefca$, of cost 28

6. (6.3.22) A knight's tour of a chessboard is a sequence of knight moves that visits each square exactly once, and returns to its starting square with just one move. Pose the knight's tour problem as one of determining whether a certain graph is hamiltonian. Give details.

   Form a graph as follows: Make 64 vertices, one for each square. Connect each vertex to all vertices corresponding to squares that are a knight's move away. A hamilton tour of this graph exactly corresponds to a knight's tour.