MATH 3330 — Applied Graph Theory
Assignment 4 — Solutions

1. True of false: the endpoints of a cut-edge are both cut-vertices. If true, explain why. If false, give a counterexample.

   False. Counterexample: Take any tree of at least two vertices. Every edge is a cut edge. However, leaves are not cut-vertices. So any edge with a leaf as endpoint is a counterexample. 1 point

2. Text, 2.4.31. Let v be one of the vertices of a connected graph G. Find an upper bound for the number of components in $G - v$ (explain your answer), and give an example that achieves that upper bound.

   The maximum number of components in $G - v$ equals the degree of v. Namely, all components of $G - v$ must contain at least one neighbour of v, because G is connected. Since the maximum degree of v equals $n - 1$, the maximum number of components in $G - v$ equals $n - 1$. Example: Let G be a start, i.e. a set of non-adjacent vertices with one central vertex adjacent to all other vertices, and let v be the central vertex. 2 points

3. (a) Text, 3.1.18. True or false: There exists a connected n-vertex simple graph with $n+1$ edges that contains exactly two cycles. If true, give an example, if false, explain why not.

   True for $n \geq 5$: take P_n, and add edges from v_1 to v_3 and v_3 to v_5.

   (b) Text, 3.1.19. True or false: There exists a connected n-vertex simple graph with $n + 2$ edges that contains four edge disjoint cycles. If true, give an example, if false, explain why not.

   False. A spanning tree of an n-vertex has $n - 1$ edges and no cycles. Every additional edge creates at least one cycle. Since there are only 3 additional edges, one edge must create more than one cycle. The only way this can happen is if the edge is a chord in an existing cycle. However, in this case the cycles are not edge-disjoint. 2 points

4. Text, 3.2.16. What is the relationship between the depth of a vertex v in a rooted tree and the number of ancestors of v? Explain your answer.

They are equal. The depth of a vertex is the distance to the root. An ancestor of v is a vertex of on the path from v to the root. Since there is only one unique path in a tree between any two vertices, there is only one path from v to the root, so this must be the shortest path. So the number of vertices (not v) on this path is the number of ancestors, and the length of this path is the depth. They must be equal. 2 points.

5. Do problems 4.2.3 and 4.2.10 of the text (find a dfs and bfs tree for the given graph).

   Since there are many different bfs and dfs trees in a graph, every student's work was judged individually. Note that the bfs tree must have height 2, since the eccentricity of x equals 2. Note also that a dfs tree cannot have cross edges.

   Example of dfs tree rooted at x (parent notation):

x	r	s	t	u	v	w	y	z
-	t	r	z	v	w	x	z	v

   Example of bfs tree rooted at x (parent notation):

x	r	s	t	u	v	w	y	z
-	x	r	z	w	w	x	x	x

   2 points.

6. Do problems 4.3.3 and 4.3.7 of the text (find an MST and a shortest-path tree for the given graph). Show your work.

   MST tree (in parent notation:

s	a	b	c	d	e	f	g	h
-	c	a	f	a	b	s	e	g

   1 point.

   (For 4.3.7, see assignment 5)