MATH 3330 — Applied Graph Theory

Assignment 6 – Solutions

1. Either draw the graph or explain why none exists:

   (a) A connected graph with 11 vertices and 10 edges and no cut-vertices.

   Does not exist. A connected graph with number of edges one less than the number of vertices is a tree, and in a tree all vertices that are not leaves are cut-vertices. Each tree with 11 vertices has at least one vertex that is not a leaf.

   (b) A 3-connected graph with exactly one bridge.

   Not possible. A graph is 3-connected if it has no vertex cut of size less than 3, or is K_4. If a graph has a bridge and more than 2 vertices, then at least one of the endpoints must be a cut vertex, i.e. a vertex cut of size 1.

   (c) A graph for which $\kappa_v(G) = \kappa_e(G) < \delta_{min}(G)$.

   Exists. For example, take two complete graphs on 5 vertices, and connect the two with one edge with an endpoint in each copy. Then the edge and vertex connectivity are 1, while the minimum degree equals 4.

2. An network engineer wants to build a network that is k-connected (for some positive integer k). She proposes the following algorithm: Start with a complete graph K_{k+1}. At each next step, add a vertex to the existing graph, and add edges from this new vertex to k existing vertices. Continue until the desired number of vertices is reached.

   Does this algorithm always produce a k-connected graph? If not, give a counterexample. If so, give an argument why.

   Yes. First note that K_{k+1} is k-connected. Now suppose that the graph is being formed as desribed above, and the first time the graph becomes not k-connected is when a certain vertex v is added. Let G be the graph after vertex v is added. By assumption, G is not k-connected, so G must have a vertex cut S of size less than k; $G - S$ thus has more than one component. But before adding v, the graph was k-connected, so $S - v$ is not a vertex cut in $G - v$. Therefore, $G - S - v$ is connected.

Therefore, one of the components of $G - S$ must consist only of v, and thus S must contain all neighbours of v. By construction, v has k neighbours, so this leads to a contradiction.

3. Consider the circulant graphs, defined in section 1.2 of the text. Give some general conditions on $n, s_1, s_2, \ldots$ for which the graph $circ(n; s_1, s_2, \ldots)$ is bipartite. Explain your answer.

   A graph is bipartite if and only if it does not have odd cycles. So $circ(n; s_1, \ldots, s_k)$ is bipartite if and only if there do not exist non-negative integers $\alpha_1, \ldots, \alpha_k$ so that $\alpha_1 + \ldots + \alpha_k$ is odd and $\alpha_1 s_1 + dots + \alpha_k s_k$ is a multiple of n.

4. Problem 6.2..2. Draw the $(2, 3)$-deBruijn digraph and use it to construct two different $(2, 3)$-deBruijn sequences.

   See handout from Crystal's presentation

5. Problem 6.2.23. Find the RNA chain that matches the given fragments.