MATH 3330 — Applied Graph Theory
Assignment 8 – Solutions

1. Do problems 13.3.4 and 13.3.12. This means: applying the max flow algorithm to determine the local vertex and edge connectivity (= size of minimum vertex or edge cut) between the two solid vertices. Your solutions should show: (1) the transformed network, and (2) all steps of the max flow algorithm.

   (13.3.4) Transform the network: replace each edge by two directed edges, one in each direction. Capacity equals one for each edge. Source is d, sink is e. Applying max flow/min cut, we find the min cut : $V_s = \{d\}$, of size 2.

   (13.3.12) Transform the network: replace each edge by two directed edges, one in each direction, and give each of these edges capacity ∞. Then split each vertex v except d and e into two vertices v_{in} and v_{out}. Replace each edge directed towards v with an edge directed towards v_{in}, and each edge going out of v by an edge going out of v_{out} (capacity remains ∞). Add edges from v_{in} to v_{out} with capacity 1. Find the minimum cut. This cut will only contain edges with capacity 1, so only edges of type $v_{in}v_{out}$. The minimum vertex cut consists of those vertices whose in-out edge is part of the minimum cut in the transformed network. In this case, min cut has size 2, and consists of $\{a, f\}$. 3 points

2. Find a maximum matching in the bipartite graph of problem 13.4.3, using the max flow algorithm. Show every step of the algorithm.

   Let X and Y be the two parts of the bipartition. Transform the network: add a source s, and edges to each vertex of X, with capacity 1. Direct all edges of the graph from X to Y, and give them capacity ∞. Add a source t, and edges from each vertex in Y to t, capacity 1.

   Find a maximum flow with the algorithm. Translate this back into a matching: the matching consists of all edges from X to Y that have flow equal to 1. 2 points

3. Solve problem 13.4.5. If the answer is yes, give an assignment of men to

women. If the answer is no, show that Hall's condition is violated(give a specific set of men and women that shows this).

The answer is yes. Possible matching: KH, BF, CJ, DG, EI. 1 point

4. (13.4.21) Let G be a graph, and let W be a subset of V_G.

   (a) Assume W is an independent set. Show that $V_G - W$ is a vertex cover.

   Let uv be an edge of G. Since W is an independent set, at most one of the endpoints u and v can be in W.Therefore, at least one endpoint must be in $V_G - W$. So every edge has at leasts one endpoint in $V_G - W$, so $V_G - W$ is a vertex cover. 1 point

   (b) Assume W is a vertex cover. Show that $V_G - W$ is an independent set.

   Let u and v be two vertices in $V_G - W$. Suppose that uv is an edge of G. Then neither of the endpoints of this edge is in W, which contradicts the fact that W is a vertex cover. So such an edge cannot exist, so $V_G - W$ is an independent set. 1 point

5. Do problem 13.5.6

   Suppose an $m \times n$ Latin rectangle is given, with $m < n$. Form a bipartite graph as follows: $X = \{c_1, \ldots, c_n\}$, $Y = \{s_1, \ldots, s_n\}$. Vertices c_i and s_j are adjacent if and only if symbol j does not occur in column i of the rectangle (and thus could be used to place in column i in the $(m+1)$-th row). If we can find a perfect matching in this graph, then we can extend the rectangle by one row, by placing the symbol matched to c_i in the i-th column of that row.

   Why can we always find a perfect matching? We have to show that Hall's condition holds. Take any set $A \subseteq X$ of size k. Every vertex in A has $n - m$ neighbours (the symbols that are not yet used in the corresponding column). So, with duplications, A has $(n - m)k$ neighbours. Each neighbour (a symbol) is adjacent to exactly $n - m$ columns (all columns in which it has not appeared yet), and thus can occur at most $n - m$ times as a duplicate. That means that there are at least $k(n - m)/(n - m) = k$ distinct neighbours, so $|N(A)| \geq |A|$. Therefore, Hall's condition holds, so there exists a matching saturating X, which must be a perfect matching since X and Y are of the same size. 2 points