Floyd-Warshall algorithm for all-pairs shortest paths

Input: Graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$, and costs $c(e)$ assigned to each edge $e \in E$.

Algorithm: Maintain a matrix M and E.

Initialize: Set M_{ij} equal to the cost of edge $v_i v_j$, if there is such an edge, ∞ otherwise. Set all entries of E to zero.

Perform the following step for $k = 1 \ldots n$. Let $M^{(k)}$ denote the matrix after k steps. Then:

$$M_{ij}^{(k)} = \min\{M_{ij}^{(k-1)}, M_{ik}^{(k-1)} + M_{kj}^{(k-1)}\}$$

If the minimum is achieved by the second term, then set $E_{ij} = k$.

In fact $M_{ij}^{(k)}$ represents the cheapest path from v_i to v_j if *only* $v_1, \ldots v_k$ *can be used as internal vertices of the path.* The calculation

$$M_{ij}^{(k)} = \min\{M_{ij}^{(k-1)}, M_{ik}^{(k-1)} + M_{kj}^{(k-1)}\}$$

shows this is so: the cheapest way to go from v_i to v_j via $v_1, ..., v_k$ has two options:

- use only $v_1, \ldots v_{k-1}$, cost $M_{ij}^{(k-1)}$, or
- use v_k, cost $M_{ik}^{(k-1)} + M_{kj}^{(k-1)}$.

$$M_{i,j}^{(1)} = \min\{M_{i,j}^{(0)}, M_{i,1}^{(0)} + M_{1,j}^{(0)}\}$$

$$M^{(0)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & & & \\ \infty & & & \\ \infty & & & \end{bmatrix}$$

Paths to and from v_1 remain the same.

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^{(1)}_{2,2} = \min\{M^{(0)}_{2,2}, M^{(0)}_{2,1} + M^{(0)}_{1,2}\}$$

$$M^{(0)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & & \\ \infty & & & \\ \infty & & & \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^{(1)}_{i,j} = \min\{M^{(0)}_{i,j}, M^{(0)}_{i,1} + M^{(0)}_{1,j}\}$$

$$M^{(0)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix}$$

All entries remain the same since there are no edges into v_1 (so no cheap paths via v_1).

$$M_{i,j}^{(2)} = \text{min}\{M_{i,j}^{(1)}, M_{i,2}^{(1)} + M_{2,j}^{(1)}\}$$

$$M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(2)} = \begin{bmatrix} & 5 & & \infty \\ \infty & \infty & -4 & 5 \\ & 5 & & \\ & 2 & & \end{bmatrix}$$

Paths through and from v_2 remain the same.

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^{(2)}_{1,1} = \min\{M^{(1)}_{1,1}, M^{(1)}_{1,2} + M^{(1)}_{2,1}\}$$

$$M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(2)} = \begin{bmatrix} \infty & 5 & & \\ \infty & \infty & -4 & 5 \\ & 5 & & \\ & 2 & & \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{1,3}^{(2)} = \mathsf{min}\{M_{1,3}^{(1)}, M_{1,2}^{(1)} + M_{2,3}^{(1)}\}$$

$$M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & \\ \infty & \infty & -4 & 5 \\ & 5 & & \\ & 2 & & \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{i,j}^{(2)} = \min\{M_{i,j}^{(1)}, M_{i,2}^{(1)} + M_{2,j}^{(1)}\}$$

$$M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 2 & 3 & \infty \end{bmatrix} \quad M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & 7 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$M_{i,j}^{(3)} = \min\{M_{i,j}^{(2)}, M_{i,3}^{(2)} + M_{3,j}^{(2)}\}$$

$$M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & 7 \end{bmatrix} \quad M^{(3)} = \begin{bmatrix} & & 1 & \\ & & -4 & \\ \infty & 5 & 1 & 1 \\ & & -2 & \end{bmatrix}$$

Paths through and from v_3 remain the same.

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$M^{(3)}_{1,1} = \min\{M^{(2)}_{1,1}, M^{(2)}_{1,3} + M^{(2)}_{3,1}\}$$

$$M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & 7 \end{bmatrix} \quad M^{(3)} = \begin{bmatrix} \infty & & 1 & \\ & & -4 & \\ \infty & 5 & 1 & 1 \\ & & -2 & \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$M_{1,2}^{(3)} = \mathrm{min}\{M_{1,2}^{(2)}, M_{1,3}^{(2)} + M_{3,2}^{(2)}\}$$

$$M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & 7 \end{bmatrix} \quad M^{(3)} = \begin{bmatrix} \infty & & 1 & 2 \\ & & -4 & \\ \infty & 5 & 1 & 1 \\ & & -2 & \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$M_{i,j}^{(3)} = \min\{M_{i,j}^{(2)}, M_{i,3}^{(2)} + M_{3,j}^{(2)}\}$$

$$M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & 7 \end{bmatrix} \quad M^{(3)} = \begin{bmatrix} \infty & 5 & 1 & 2 \\ \infty & 1 & -4 & -3 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$$M^{(3)} = \begin{bmatrix} \infty & 5 & 1 & 2 \\ \infty & 1 & -4 & -3 \\ \infty & 5 & 1 & 1 \\ \infty & 2 & -2 & -1 \end{bmatrix}$$

A negative entry on the diagonal means there is a negative cost path from a vertex to itself, so a negative cost cycle. This means all the other information is false, and we can stop the process right here.

Another example:

$$M_{i,j}^{(1)} = \min\{M_{i,j}^{(0)}, M_{i,1}^{(0)} + M_{1,j}^{(0)}\}$$

$$M^{(0)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 4 & 3 & \infty \end{bmatrix} \quad M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 4 & 3 & \infty \end{bmatrix}$$

$$M_{i,j}^{(2)} = \min\{M_{i,j}^{(1)}, M_{i,2}^{(1)} + M_{2,j}^{(1)}\}$$

$$M^{(1)} = \begin{bmatrix} \infty & 5 & 3 & \infty \\ \infty & \infty & -4 & 5 \\ \infty & 5 & \infty & 1 \\ \infty & 4 & 3 & \infty \end{bmatrix} \quad M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 4 & 0 & 9 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$M_{i,j}^{(3)} = \mathrm{min}\{M_{i,j}^{(2)}, M_{i,3}^{(2)} + M_{3,j}^{(2)}\}$$

$$M^{(2)} = \begin{bmatrix} \infty & 5 & 1 & 10 \\ \infty & \infty & -4 & 5 \\ \infty & 5 & 1 & 1 \\ \infty & 4 & 0 & 9 \end{bmatrix} \quad M^{(3)} = \begin{bmatrix} \infty & 5 & 1 & 2 \\ \infty & 1 & -4 & -3 \\ \infty & 5 & 1 & 1 \\ \infty & 4 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$$M^{(4)}_{i,j} = \min\{M^{(3)}_{i,j}, M^{(3)}_{i,4} + M^{(3)}_{4,j}\}$$

$$M^{(3)} = \begin{bmatrix} \infty & 5 & 1 & 2 \\ \infty & 1 & -4 & -3 \\ \infty & 5 & 1 & 1 \\ \infty & 4 & 0 & 1 \end{bmatrix} \quad M^{(4)} = \begin{bmatrix} \infty & 5 & 1 & 2 \\ \infty & -1 & -4 & -3 \\ \infty & 5 & 1 & 1 \\ \infty & 4 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

Identify the cheapest path from v_1 to v_4 using E:

$$E = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$$\begin{array}{ll} v_1 \rightarrow v_4 & E_{1,4} = 2 \\ v_1 \rightarrow v_2 \rightarrow v_4 & E_{2,4} = 3 \\ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 & E_{1,2} = 0 \\ v_1 - v_2 \rightarrow v_3 \rightarrow v_4 & E_{2,3} = 0 \\ v_1 - v_2 - v_3 \rightarrow v_4 & E_{3,4} = 0 \\ v_1 - v_2 - v_3 - v_4 & \end{array}$$