MATH 5330 ASSIGNMENT 6

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Question 1

Solution: Let G be a k-regular graph of size n, so every vertex has degree k.

(a) We claim that $\chi_l(G) \leq k+1$. To see this, assign lists of size $k+1$ to each vertex of $V(G)$, let $V(G)$ be ordered with any order and colour greedily. When colouring v_i in this greedy colouring there will be at most k colours used in the neighbourhood of v_i and since v_i was assigned a list of size $k+1$, there will be at least one colour remaining in $v_i's$ list that v_i can be coloured with. Therefore G is $(k+1)$-choosable so $\chi_l(G) \leq k+1$.

(b) Suppose $L : V(G) \to \mathcal{P}A$ is a list assignment of G so that $|L(v)| = k$ for all $v \in V(G)$. Order $V(G)$ at random and apply the greedy algorithm. By this we mean for each v_i if there is a colour in its list that can be used to colour v_i, then colour it, otherwise leave it uncoloured. Define the random variable X to be the number of vertices that can be coloured in this way. For each $v \in V(G)$ define the indicator variable X_v as follows:

$$X_v = \begin{cases} 1 & \text{if } v \text{ can be coloured} \\ 0 & \text{otherwise} \end{cases}$$

Now $E(X_v) = \mathbb{P}(X_v = 1) \geq \mathbb{P}(X_v$ is ordered before at least one of its k neighbours$) \geq \frac{k}{k+1}$. So the linearity of expectation gives:

$$\begin{aligned} E(X) &= \sum_{v \in V(G)} E(X_v) \\ &\geq \sum_{v \in V(G)} \frac{k}{k+1} \\ &= n \cdot \frac{k}{k+1} \end{aligned}$$

So by the pigeonhole property of expectation, there exists an ordering of $V(G)$ such that there exists a list colouring under L of at least $n \cdot \frac{k}{k+1}$ vertices of G.

(c) Suppose $1 \leq t \leq k$. Suppose $L : V(G) \to \mathcal{P}A$ is a list assignment of G so that $|L(v)| = t$ for all $v \in V(G)$. Order $V(G)$ at random and apply the greedy algorithm. Define the random variable X to be the number of vertices that can be coloured with the greedy algorithm. For each $v \in V(G)$ define the indicator variable X_v as follows:

$$X_v = \begin{cases} 1 & \text{if } v \text{ can be coloured} \\ 0 & \text{otherwise} \end{cases}$$

Now $E(X_v) = \mathbb{P}(X_v = 1) \geq \mathbb{P}(X_v \text{ is ordered before at least } k - t \text{ of its } k \text{ neighbours}) \geq \frac{t}{k+1}$. So the linearity of expectation gives:

$$\begin{aligned} E(X) &= \sum_{v \in V(G)} E(X_v) \\ &\geq \sum_{v \in V(G)} \frac{t}{k+1} \\ &= n \cdot \frac{t}{k+1} \end{aligned}$$

So by the pigeonhole property of expectation, there exists an ordering of $V(G)$ such that there exists a list colouring under L of at least $n \cdot \frac{t}{k+1}$ vertices of G.

Question 2

Solution: Consider $G = G(n, p)$. Let X be the number of edges in G.

(a) For all subsets S such that $|S| = 2$ of $V(G)$ define the indicator variable X_S as follows:

$$X_S = \begin{cases} 1 & \text{if } G[S] \cong K_2 \\ 0 & \text{otherwise} \end{cases}$$

I.e, $X_S = 1$ if the vertices in S are joined by an edge. Now $X = \sum_{S \subseteq V(G),\ |S|=2} X_S$, so by the linearity of expectation,

$$E(X) = \sum_{S \subseteq V(G),\ |S|=2} E(X_S).$$

(a) We want to find $E(X_S) = \mathbb{P}(X_S = 1)$, which by definition of $G(n,p)$ is p. So from part (a), we have:

$$\begin{aligned} E(X) &= \sum_{S \subseteq V(G),\ |S|=2} E(X_S) \\ &= \sum_{S \subseteq V(G),\ |S|=2} p \\ &= \binom{n}{2} p \end{aligned}$$

(b) Markov's inequality gives:

$$\mathbb{P}(X \geq n) \leq \frac{E(X)}{n} = \frac{1}{n} \cdot \binom{n}{2} p$$

(c) Let p be a function of n, and assume that $p = o(\frac{1}{n})$. Now by (b), we know that

$$\begin{aligned} \mathbb{P}(X \geq n) &\leq \frac{1}{n} \cdot \binom{n}{2} p \\ &\leq np \\ &\to 0 \text{ as } n \to \infty \text{ (Since } p = o(\frac{1}{n})) \end{aligned}$$

So a.a.s. $e(G) \leq n-1$ and since a graph on n vertices must have at least $n-1$ edges, a.a.s G is not connected.

(d) By definition of $G(n,p)$, the variables $X_S's$ are independent so $Cov(X_S, X_T) = 0$ if $S \neq T$ and therefore we have linearity of variance. Also, $Var(X_S) = E((X_S)^2) -$

$E(X_S)^2 = p - p^2$. Therefore,

$$\begin{aligned} Var(X) &= Var\Big(\sum_{S\subseteq V(G),\ |S|=2} X_S\Big) \\ &= \sum_{S\subseteq V(G),\ |S|=2} \Big(Var X_S\Big) \\ &= \sum_{S\subseteq V(G),\ |S|=2} (p-p^2) \\ &= \binom{n}{2} p(1-p) \\ &= E(X)(1-p) \end{aligned}$$

(e) We first note that $n-1 = \frac{2E(X)}{np}$. Now,

$$\begin{aligned} \mathbb{P}(X \geq n) &= 1 - \mathbb{P}(X \leq n-1) \\ &\geq 1 - \mathbb{P}\Big(|X - E(X)| \geq \frac{(2-np)E(X)}{np}\Big) \\ &\geq 1 - \frac{(np)^2 Var(X)}{(np-2)^2 E(X)^2} \text{ (by Chebeshev's inequality applied to } \mathbb{P}(X \leq n-1)) \\ &= 1 - \frac{n^2p^2(1-p)}{(np-2)^2 E(X)} \text{ (By part (d))} \end{aligned}$$

(f) Now, if p is constant, we see that:

$$\begin{aligned} \mathbb{P}(X \geq n) &\geq 1 - \frac{n^2p^2(1-p)}{(np-2)^2E(X)} \text{ (By part (e))} \\ &= 1 - \frac{2p(1-p)}{(p-\frac{2}{n})^2(n(n-1))} \\ &\to 1 - 0 = 1 \text{ as } n \to \infty \text{ since } p \text{ is constant} \end{aligned}$$

Therefore, $\mathbb{P}(X \geq n) \to 1$ as $n \to \infty$. Also, since a graph with more than $n-1$ edges must contain a cycle, a.a.s. G contains a cycle.

Question 3

Solution: Consider the graph model $G = G(n, p)$.

(a) Let X be the random variable counting the number of triangles. For all subsets S such that $|S| = 3$ of $V(G)$ define the indicator variable X_S as follows:

$$X_S = \begin{cases} 1 & \text{if } G[S] \cong K_3 \\ 0 & \text{otherwise} \end{cases}$$

Now, $E(X_S) = \mathbb{P}(X_S = 1) = p^3$, so by the linearity of expectation,

$$\begin{aligned} E(X) &= \sum_{S \subseteq V(G),\ |S|=3} E(X_S) \\ &= \sum_{S \subseteq V(G),\ |S|=3} p^3 \\ &= \binom{n}{3} p^3 \end{aligned}$$

(b) $Var(X) = Var(\sum X_S) = \sum_{s \subseteq V(G):\ |S|=3} Var(X_S) + \sum_{S \neq T} Cov(X_S, X_T)$ (*). Now $Var(X_S) = E(X_S^2) - (E(X_S))^2 = p^3 - p^6 \leq p^3$. Also, for $S \neq T$,

$$\begin{aligned} Cov(X_S, X_T) &= E(X_S X_T) - E(X_S)E(X_T) \\ &= \mathbb{P}(X_S X_T = 1) - p^6 \\ &\leq p^5 - p^6 \quad \text{(since 2 triangles may share at most 1 edge)} \\ &\leq p^5 \end{aligned}$$

So from (*),

$$\begin{aligned} Var(X) &\leq \sum_{s \subseteq V(G):\ |S|=3} p^3 + \sum_{S \neq T} p^5 \\ &\leq \binom{n}{3} p^3 + 4 \cdot \binom{n}{4} p^5 \quad \text{(since every subset of 4 vertices contains 4 possible triangles)} \\ &\leq n^3 p^3 + n^4 p^4 \quad \text{(since } p \leq 1\text{)} \end{aligned}$$

(c) Claim: The threshold function for G has a triangle is $f(n) = \frac{1}{n}$.

i) If $\frac{1}{n} \ll p$, then by Chebyshev's inequality:

$$\begin{aligned} \mathbb{P}(X = 0) &\leq \mathbb{P}(|X - E(X)| \geq E(X)) \\ &\leq \frac{Var(X)}{E(X)^2} \\ &\leq \frac{n^3 p^3 + n^4 p^4}{(\binom{n}{3} p^3)^2} \\ &= \theta(\frac{1}{n^3 p^3} + \frac{1}{n^2 p^2}) \\ &\to 0 \text{ as } n \to \infty \text{ (since } \frac{1}{n} \ll p) \end{aligned}$$

ii) If $p \ll \frac{1}{n}$, then by Markov's inequality:

$$\begin{aligned} \mathbb{P}(X \geq 1) &\leq E(X) \\ &\leq (np)^3 \\ &\to 0 \text{ as } n \to \infty \text{ (since } p \ll \frac{1}{n}) \end{aligned}$$

Therefore, $f(n) = \frac{1}{n}$ is a threshold function for "G has a triangle".

Question 5

For every graph H there exists a function $p = p(n)$ so that $\lim_{n \to \infty} p(n) = 0$ but a.a.s a graph G produced by $G(n, p)$ contains an induced copy of H.

Proof. We prove the result by the probabilistic method. Fix H, a finite graph. Let X be the number of induced subgraphs of $G(n,p)$ of size $|H|$ that are isomorphic to H. We define the indicator variable X_S for all $S \subseteq V(G)$ with $|S| = |H|$ as follows:

$$X_S = \begin{cases} 1 & \text{if } G[S] \cong H \\ 0 & \text{otherwise} \end{cases}$$

Let $m = \max\{e(H), \binom{|H|}{2} - e(H)\}$. We claim that if $p(n) = (\frac{1}{n})^{\frac{|H|-1}{m}}$, then a.a.s G contains an induced copy of H. Now $X = \sum_{S \subseteq V(G):|S|=|H|} X_s$ and $E(X_S) = P(X_S = 1) \geq p^{e(H)}(1-p)^{\binom{|H|}{2}-e(H)}$, so

$$\begin{aligned} E(X) &= E\Big(\sum_{S\subseteq V(G):|S|=|H|} X_S\Big) \\ &= \sum_{S\subseteq V(G):|S|=|H|} E(X_S) \\ &\geq \sum_{S\subseteq V(G):|S|=|H|} p^{e(H)}(1-p)^{\binom{|H|}{2}-e(H)} \\ &= \binom{n}{|H|} p^{e(H)}(1-p)^{\binom{|H|}{2}-e(H)} \\ &\geq \Big(\frac{n}{|H|}\Big)^{|H|} p^m (1-p)^m \quad \text{since } p, (1-p) < 1 \\ &= \Big(\frac{n}{|H|}\Big)^{|H|} \Big(\frac{1}{n}\Big)^{\frac{m(|H|-1)}{m}} (1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^m \\ &= \Big(\frac{n}{|H|^{|H|}}\Big)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^m \end{aligned}$$

And since $|H|$ and m are constants, $(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^m \to 1$ as $n \to \infty$, so,

$$\Big(\frac{n}{|H|^{|H|}}\Big)\Big(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}}\Big)^m \to \infty \text{ as } n \to \infty.$$

Therefore, $E(X) \to \infty$ as $n \to \infty$.

We now compute the variance of X so that we may apply Chebyshev's Inequality to show that $\mathbb{P}(X = 0) \to 0$ as $n \to \infty$.

$$\begin{aligned}
Var(X) &= \sum_{S\subseteq V(G):\ |S|=|H|} Var(X_S) + \sum_{S\neq T} Cov(X_S, X_T) \\
&= \sum_{S\subseteq V(G):\ |S|=|H|} \Big(E(X_S^2) - E(X_S)^2\Big) + \sum_{S\neq T} \Big(E(X_S X_T) - E(X_S)E(X_T)\Big) \\
&\leq E(X) - \Big(\frac{n}{|H|^{|H|}}\Big)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^{2m} + \sum_{S\neq T}(E(X_S X_T)) - \left(\binom{\binom{n}{|H|}}{2}\right)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^{2m} \\
&\leq 2E(X) - \Big(\frac{n}{|H|^{|H|}}\Big)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^{2m} \ (*)
\end{aligned}$$

Now, by Chebyshev,

$$\begin{aligned}
\mathbb{P}(X = 0) &\leq \mathbb{P}(|E(X) - X| \geq E(X)) \\
&\leq \frac{Var(X)}{E(X)^2} \\
&\leq \frac{2E(X) - \Big(\frac{n}{|H|^{|H|}}\Big)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^{2m}}{E(X)^2} \quad \text{(by (*))} \\
&\leq \frac{2}{E(X)} - \frac{\Big(\frac{n}{|H|^{|H|}}\Big)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^{2m}}{E(X)^2} \\
&\to 0 - 0 = 0 \ \text{ as } n \to \infty \text{ (since } E(X) \to \infty \text{ and } \Big(\frac{n}{|H|^{|H|}}\Big)(1 - \Big(\frac{1}{n}\Big)^{\frac{|H|-1}{m}})^{2m} < E(X)
\end{aligned}$$

Therefore, $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) \to 1$ as $n \to \infty$. So a.a.s. G contains a copy of H.

□