Topics in Graph Theory – Problem set 5

Due Tuesday, Feb. 25, beginning of class

1. A *dominating set* of a graph $G = (V, E)$ is a set $A \subseteq V$ so that for each vertex $v \in V$, either v is in A or v has a neighbour in A. Let $\gamma(G)$ denote the size of the smallest dominating set of G. A *doubly indenpendent set* is a set $B \subseteq v$ so that no two vertices in B have a common neighbour. In other words, the graph distance between any two vertices in B is at least three. Let $\alpha_2(G)$ denote the size of the largest doubly indenpendent set of G.

   (a) Find γ and α_2 for the circulant graphs $C(n, k)$.

   (b) Show that, for all graphs G, $\alpha_2(G) \leq \gamma(G)$. Show also that this implies that, if a graph G has a dominating set and a doubly independent set of equal size, then both are optimal.

   (c) Show that for all paths P_n, $\gamma(P_n) = \alpha_2(P_n)$.

2. In class, we saw the *binary* random graph model, $G(n, p)$. There is another random graph model, the *uniform* random graph model, $G(n, M)$,defined as follows: Given n nodes, add exactly M edges to the graph at random.

   (a) Give the formal probability space $(\Omega, \mathcal{F}, \mathbb{P})$ corresponding to this model.

   (b) Let $n = 5$. Calculate the probability (for any M) that of the event "G has a cycle".

3. Consider the Gallai-Roy-Vitaver theorem presented by Kyle, which states that, if G has a orientation where $I(G)$ is the length of the longest path, then $\chi(G) \leq I(G)$. Can we also conclude that $\chi_\ell(G) \leq I(G)$? Motivate your answer.

4. Consider the Erdös-Rubin-Taylor result presented by Emma. Use the result to obtain a lower bound k on the list chromatic number $\chi_\ell(K_{10,10})$. Give the specific unfeasible list assignment with lists of size $k-1$ which shows that the minimum size of the lists has to be at least k.

5. Consider graphs G formed according to the model $G(n, p)$ where $p = o(1)$. Show that, *asymptotically almost surely*, G is a forest (so G has no cycles). *Partially done in class. Consider the variable X counting the number of cycles, show that $E(X)$ is small, then use Markov's inequality..*

6. [MATH 5330] Show that any graph G contains a bipartite subgraph (not necessarily induced!) which contains at least half of the edges. Use the probabilistic method.

7. [MATH 4330/CSCI 4115] Give a formula for the expected number of isolated vertices (vertices of degree 0) in the random graph $G(n, p)$ where $p \in (0, 1)$ is a constant. Give the asymptotic behaviour of this expected number *What happens to the expectation when $n \to \infty$.*