Topics in Graph Theory – Problem set 6

Due Tuesday, March 4, beginning of class

1. Let G be a k-regular graph of size n, so every vertex has degree k.
   (a) Show that $\chi_\ell(G) \leq k+1$. *No probability required.*
   (b) Suppose a list of size k is assigned to each vertex of G. Now consider the problem of colouring as many vertices as possible with colours from their list. Show, using a probabilistic argument, that at least $(\frac{k}{k+1})n$ vertices can be coloured. *Hint: use greedy colouring using a random ordering. Review the argument for finding an independent set using a random ordering given in class.*
   (c) Extend the argument to show that, with lists of size t, $1 \leq t \leq k$, at least $(\frac{t}{k+1})n$ vertices can be coloured.

2. Consider $G(n,p)$. Let X be the number of edges in G. (a) Find $E(X)$ using indicator variables. (a) Find the expected value of X, in terms of n and p.
   (b) Use Markov's inequality to give an upper bound for $P(X \geq n)$.
   (c) Let p be a function of n, and assume that $p = o(1/n)$. Use the result from (b) to conclude that *a.a.s.* the graph is not connected. *Use and state a basic result about the number of edges in a graph and connectedness.*
   (d) Find the variance of X, in terms of n and p. Use the variance and covariance of the indicator variables.
   (e) Use Chebyshev's inequality to give a lower bound for $P(X \geq n)$.
   (f) Let p be a constant. Use the result from (e) to conclude that *a.a.s.* the graph contains a cycle. *Use and state a basic result about the number of edges and the existence of a cycle in a graph.*

3. Consider the graph model $G(n,p)$.
   (a) Let X be the random variable counting the number of triangles. Find the expected value of X, in terms of n and p.
   (b) Find the variance of X, in terms of n and p. Use the variance and covariance of the indicator variables.
   (c) Consider p as a function of n. Find a threshold for the property "G has a triangle". I.e. find a function $f(n)$ such that, if $p << f(n)$, then the probability that G produced by $G(n,p)$ has a triangle goes to zero as n goes to infinity, and if $p >> f(n)$ then the probability that G has

a triangle goes to one. Use Markov's and Chebyshev's inequalities to prove your result.

4. [MATH 4330, CSCI 4115] Consider $G(n,p)$, with $p \in (0,1)$ constant. Show that *a.a.s.* a graph G produced by $G(n,p)$ has diameter 2.

5. [MATH 5330] Show that for every graph H there exists a function $p = p(n)$ so that $\lim_{n\to\infty} p(n) = 0$ but *a.a.s* a graph G produced by $G(n,p)$ contains an induced copy of H.