Dominating Sets

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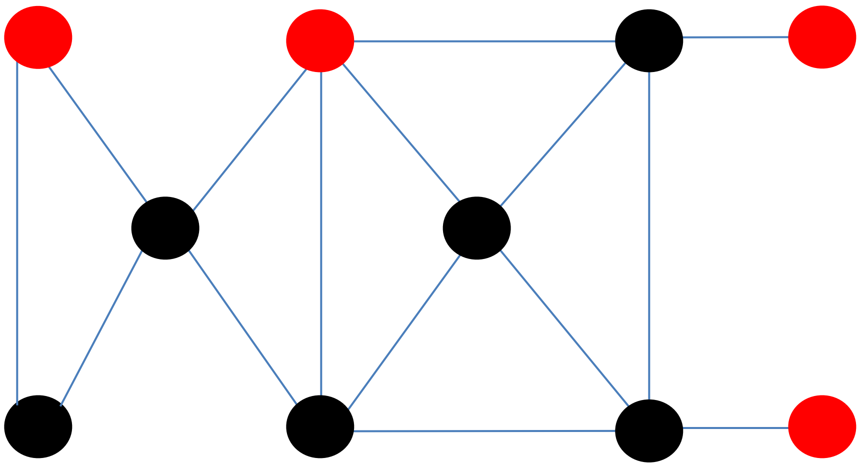
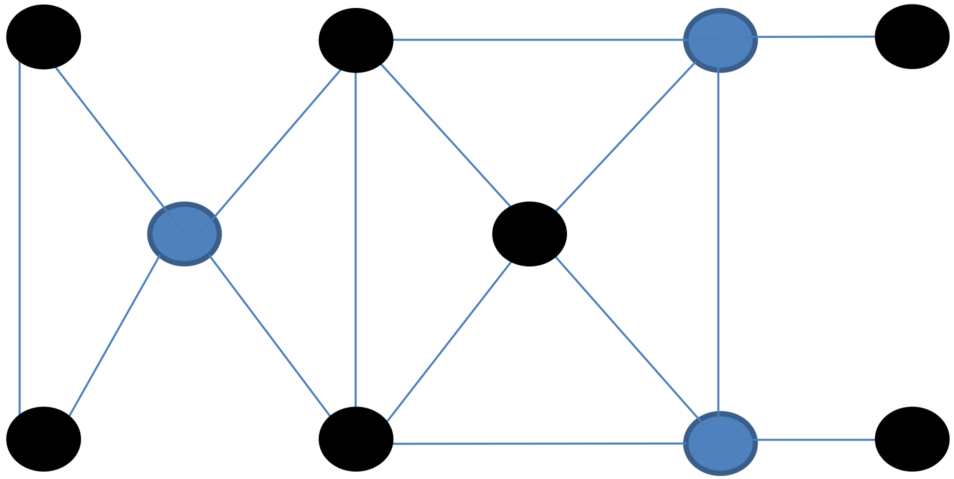
Definition

- In a graph G, a set $S \subseteq V(G)$ is a **dominating set** if every vertex not in S has a neighbour in S.
- The **domination number** $\boldsymbol{\gamma(G)}$ is the minimum size of a dominating set in G.



Minimal dominating set of size 4

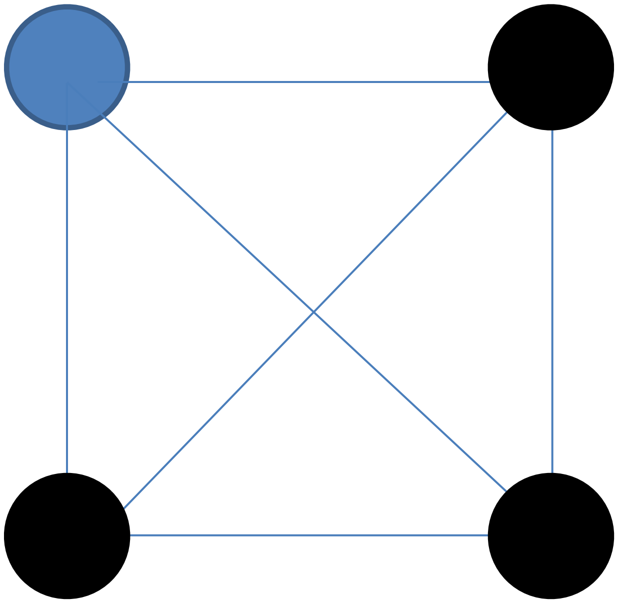
Minimum dominating set of size 3 → $\gamma(G) = 3$



- Note:
  $\beta(G) = minimum\ size\ of\ a\ vertex\ cover.$
- When a graph G has no isolated vertices, every vertex cover is a dominating set, so $\gamma(G) \leq \beta(G)$.

The difference can be large; $\gamma(k_n) = 1$, but $\beta(k_n) = n - 1$

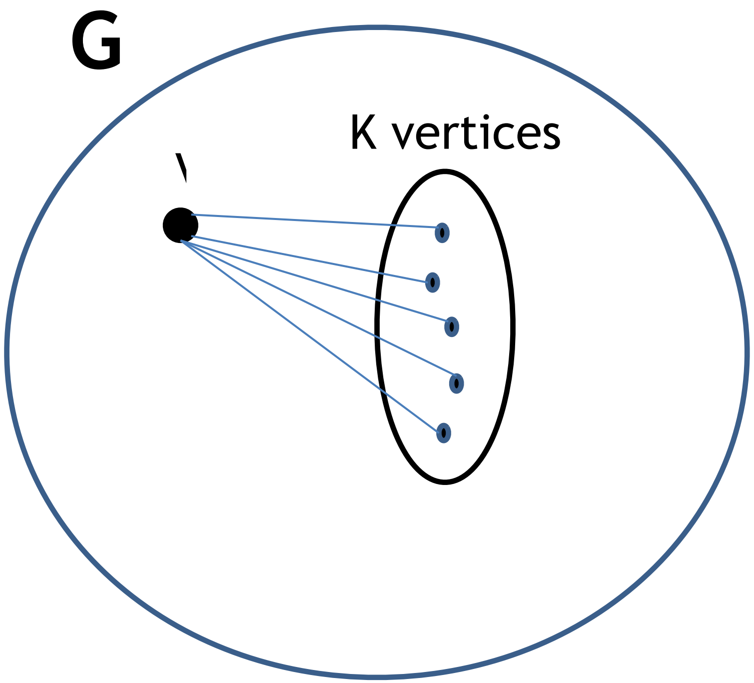
Example: K_4



$\gamma(k_4) = 1$

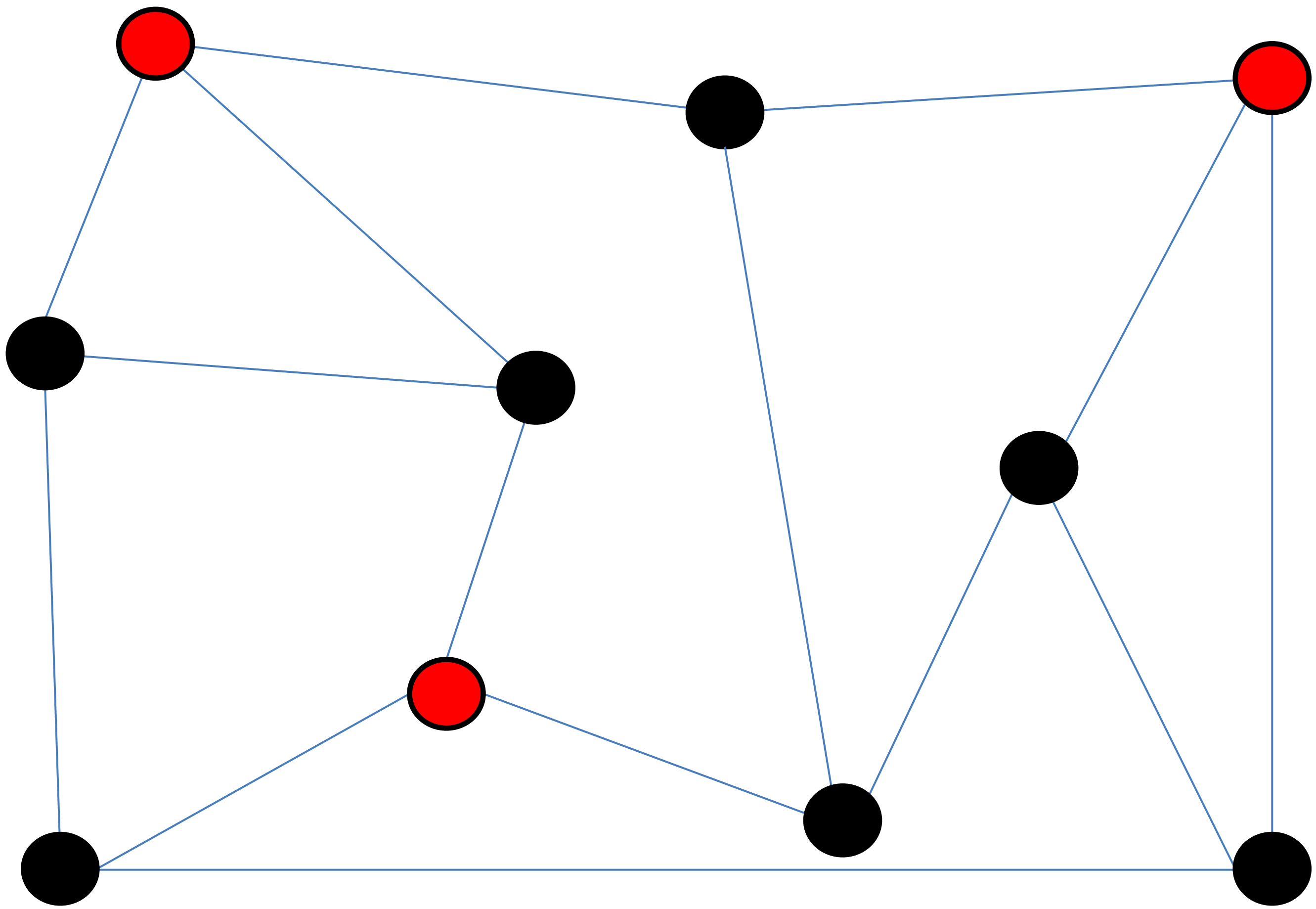
$\beta(k_4) = 4 - 1 = 3$

- A vertex of degree k dominates itself and k other vertices.



➡ Every dominating set in a k-regular graph G has size at least $\frac{n(G)}{k+1}$

➡ For every graph with minimum degree k, a greedy algorithm produces a dominating set not too much bigger than this.



3-regular graph G

$$\frac{n(G)}{k+1} = \frac{10}{3+1} = 2.5$$

So each dominating set has to have at least 3 vertices.

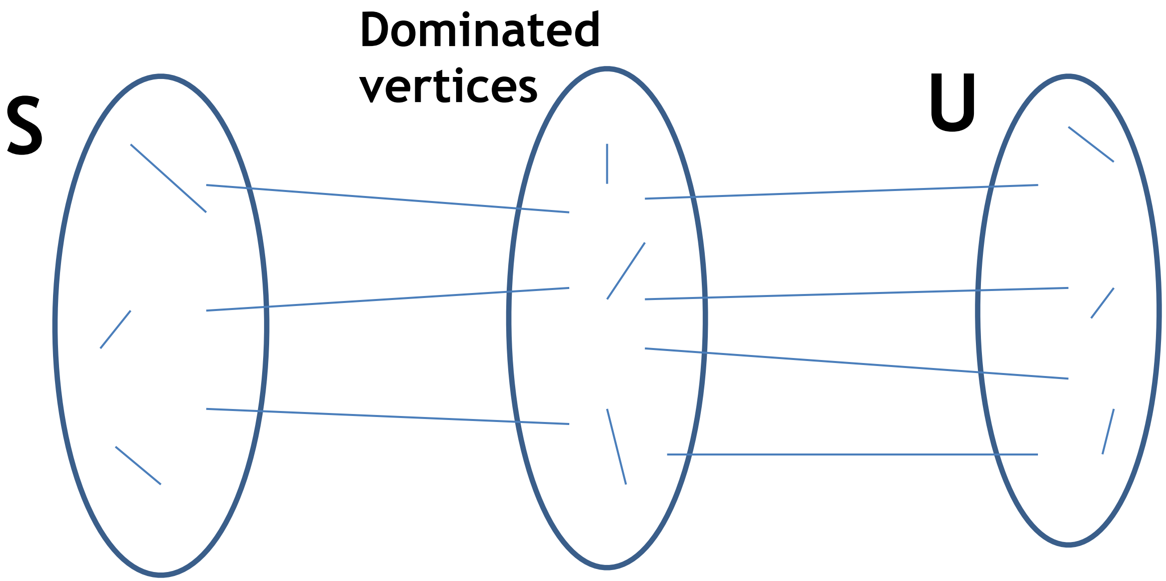
Theorem (Arnautov 1974/Payan 1975)

- Every n-vertex graph with minimum degree k has a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$.

i.e Let S be a set of vertices that form a dominating set, then $|S| \leq n\frac{1+\ln(k+1)}{k+1}$

Proof (Alon 1990)

- • Let G be a graph with minimum degree k
  - Given $S \subseteq V(G)$, let U be the set of vertices not dominated by S



Claim: There is a vertex y not in S, that dominates at least $\frac{|U|(k+1)}{n}$ vertices of U.

Each vertex in U has at least k neighbours, so $\sum_{v \in U} |N[v]| \geq |U|(k+1)$

Each vertex in G is counted at most n times by these $|U|$ sets, so some vertex y appears at least $\frac{|U|(k+1)}{n}$ times and satisfies the claim.

We iteratively select a vertex that dominates the most of the remaining undominated vertices.

We have proved that when r undominated vertices remain, after the next selection at most $r\left(1-\frac{k+1}{n}\right)$ undominated vertices remain.

Hence after $\mathrm{n}\frac{\ln(k+1)}{(k+1)}$ steps the number of undominated vertices is at most

$$n\left(1-\frac{k+1}{n}\right)^{nln(k+1)/(k+1)} < ne^{-\ln(k+1)} = \frac{n}{k+1}$$

The selected vertices and these remaining undominated vertices together form a dominating set.

Size of dominating set= # selected vertices from the iteration
+ # of vertices left in U (undominated)

$$\leq n\frac{\ln(k+1)}{k+1}+\frac{n}{k+1}$$

$$\leq n\left(\frac{\ln(k+1)+1}{k+1}\right)$$