Geographical threshold graphs (GTGs)

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Construction

The GTG model is constructed from a set of n nodes placed independently and uniformly at random in a volume in $\mathbb{R}^d$. A nonnegative weight w_i, taken randomly and independently from a probability distribution function $f(w) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is assigned to each node v_i for $i \in [n]$. Let $F(x) = \int_0^x f(w)\,dw$ be the cumulative density function. For two nodes v_i and v_j at distance r, the edge (i, j) exists if and only if the following connectivity relation is satisfied:

$$G(w_i, w_j)h(r) \geq \theta_n, \qquad (2.1)$$

where θ_n is a given threshold parameter that depends on the size of the network. The function $h(r)$ specifies the connection probability as a function of distance and is assumed to be decreasing in r. In the following we take $h(r) = r^{-\beta}$, for some positive β, which is typical, for example, of the path-loss model in wireless networks [Bradonjić and Kong 07]. The interaction strength between nodes $G(w_i, w_j)$ is typically taken to be symmetric (to produce an undirected graph) and either multiplicatively or additively separable, i.e., in the form of $G(w_i, w_j) = g(w_i)g(w_j)$ or $G(w_i, w_j) = g(w_i) + g(w_j)$.

Example

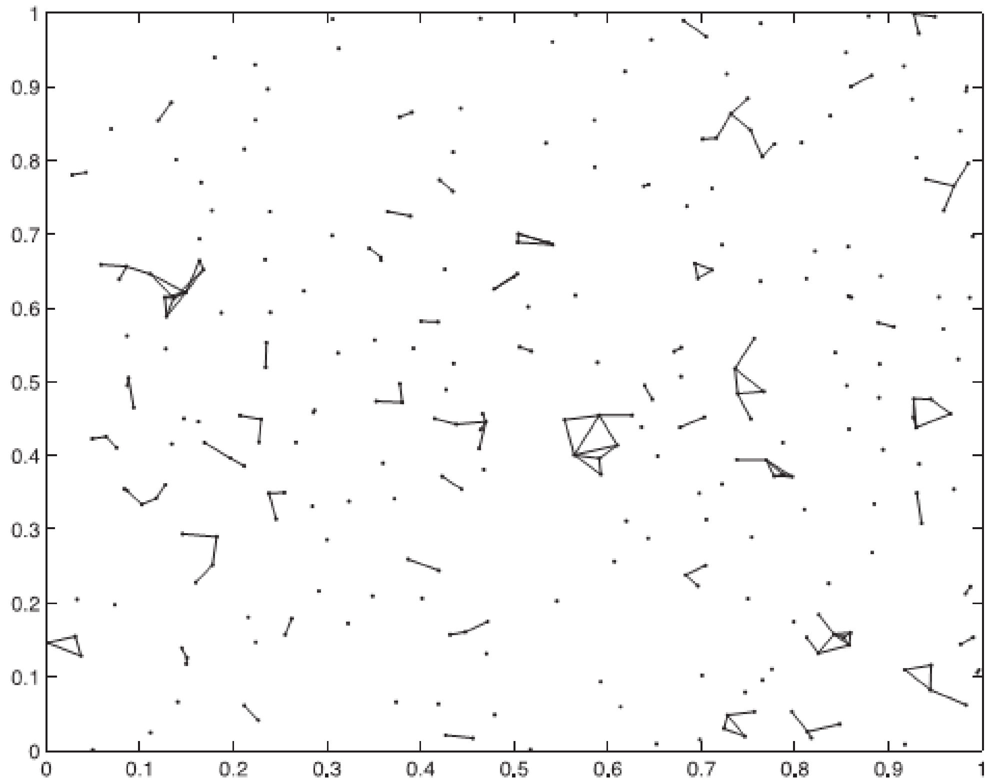
Here we restrict ourselves to the case of $g(w) = w$, $\beta = 2$, and nodes distributed uniformly over a two-dimensional space. For analytical simplicity we take the space to be a unit torus, and use the additive model for the connectivity relation

$$\frac{w_i + w_j}{r^2} \geq \theta_n. \tag{2.2}$$

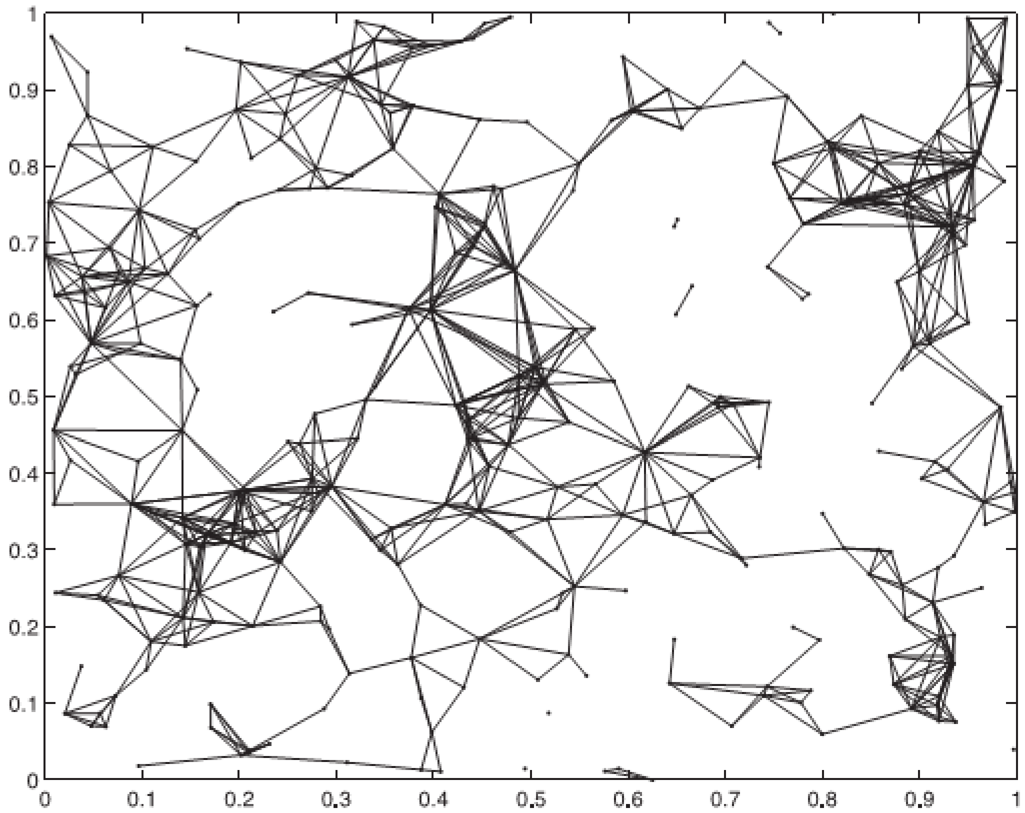
Example

Some examples of GTG instances with exponential weight distribution $f(w) = e^{-w}$ are shown in Figure 1.

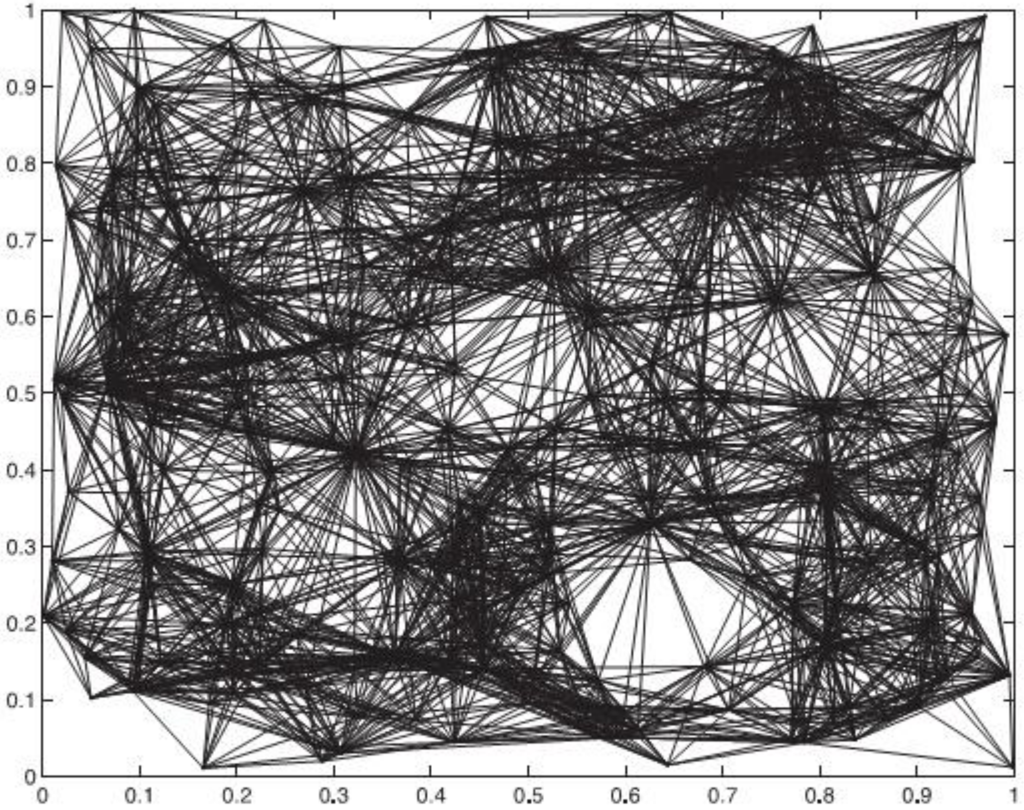
Figure 1. Instances of GTG with exponential weight distribution for $n = 300$ at decreasing threshold parameter values (increasing mean degree): (a) $\theta_n/n = 2\pi$, well below the percolation transition; (b) $\theta_n/n = 1$, above the percolation but below the connectivity transition; (c) $\theta_n/n = 1/2e$, well above connectivity.



(a)



(b)



(c)

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Degree distribution

That is, the degree distribution d_i of a node v_i with weight w_i follows the binomial distribution

$$d_i(\cdot|w_i) \sim \mathrm{Bin}(n-1, p_i), \tag{3.2}$$

where

$$p_i = \frac{\pi}{\theta_n}(w_i + \mu). \tag{3.3}$$

Giant Component

Definition 4.1. (Giant Component.) The *giant component* is a connected component of size $\Theta(n)$.

In this section we analyze the conditions for the existence of the giant component, giving bounds on the threshold parameter value θ_n where it first appears.

Absence of Giant Component

Theorem 4.2. *Let* $\theta_n = gn$ *for* $g > g'$, *where* $g' = 2\pi\mu$. *Then whp there is no giant component in GTG.*

whp : with high probability

Existence of Giant Component

Theorem 4.3. *Let* $\theta_n = gn$ *for* $g < g'' = \sup_{\alpha\in(0,1)} \alpha F^{-1}(1-\alpha)/\lambda_c$, *where* $\pi\lambda_c \approx 4.52$ *is the mean degree at which the giant component first appears in random geometric graphs (RGG) [Penrose 03]. Then whp the giant component exists in GTG.*

Gap between two bounds

Claim 4.4. *For any weight distribution* $f(w)$, $g'/g'' \geq 2\pi\lambda_c \approx 9.04$.

Remark 4.5. For the exponential distribution $f(w) = \gamma \exp(-\gamma w)$, we have $g' = 2\pi/\gamma$.

Remark 4.6. If $\alpha F^{-1}(1-\alpha)$ has an extremum for $\alpha \in (0,1)$, this occurs at

$$\alpha = F^{-1}(1-\alpha) f(F^{-1}(1-\alpha)).$$

For example, for the exponential distribution the maximum is at $\alpha = 1/e$, giving a bound of $g'' = 1/e\gamma\lambda_c$.

Connectivity

Definition 5.1. (Connectivity.) The graph on n vertices is *connected* if the largest component has size n.

In this section we analyze conditions for connectivity, giving bounds on the threshold parameter θ_n at which the entire graph first becomes connected. Sim-

Disconnected Graph

Theorem 5.2. *Let* $\theta_n = \kappa n / \log n$ *for* $\kappa > \kappa'$, *where* $\kappa > \pi\mu$. *Then the GTG is disconnected whp.*

Connected Graph

Theorem 5.3. *Let* $\theta_n = \kappa n / \log n$ *for* $\kappa < \sup_{\alpha \in (0,1)} \alpha F^{-1}(1-\alpha)/4$. *Then the GTG is connected whp.*

Diameter

Lemma 6.1. *Let the cumulative weight distribution function be $F(w)$ in the GTG model. Let x and a sequence $s_n = \Theta(x^2\theta_n)$ be such that*

$$\lim_{n\to\infty} \left(1 - F(s_n)^{nx^2/2}\right)^{1/x} = 1. \tag{6.1}$$

Then whp, $\text{diam} = O(1/x)$.

Some class of Diamter

- *Ultralow Latency:* diam $= O(1)$. Let $x < 1$ be a constant and $s_n = \theta_n$. If $F(\theta_n)^n \to 0$, then diam $= O(1)$ whp. For the exponential weight distribution it follows that $\theta_n = o(\log n)$.

- *Low Latency:* diam $= O(\log^q n)$. Let $x = 1/\log^q n$ and $s_n = \theta_n/\log^{2q} n$. If $F(\theta_n/\log^{2q} n)^{n/(2\log^{2q} n)} \log^q n \to 0$, then diam $= O(\log^q n)$ whp. For the exponential weight distribution it follows that

$$\theta_n = o\Big((\log n)^{2q(1-(\log^{2q} n)/n)}\Big).$$

- *High Latency:* diam $= O(\sqrt{n/\log n})$. Let $x = \sqrt{\log n/n}$ and $s_n = \theta_n \log n/n$. If $\sqrt{n/\log n} F(\theta_n \log n/n)^{\log n} \to 0$, then diam $= O(\sqrt{n/\log n})$ whp. For the exponential weight distribution it follows that

$$\theta_n = o\Big((n/\log n)^{1-1/(2\log n)}\Big).$$