The Small World Problem: An Algorithmic Perspective

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Local Vertex Information

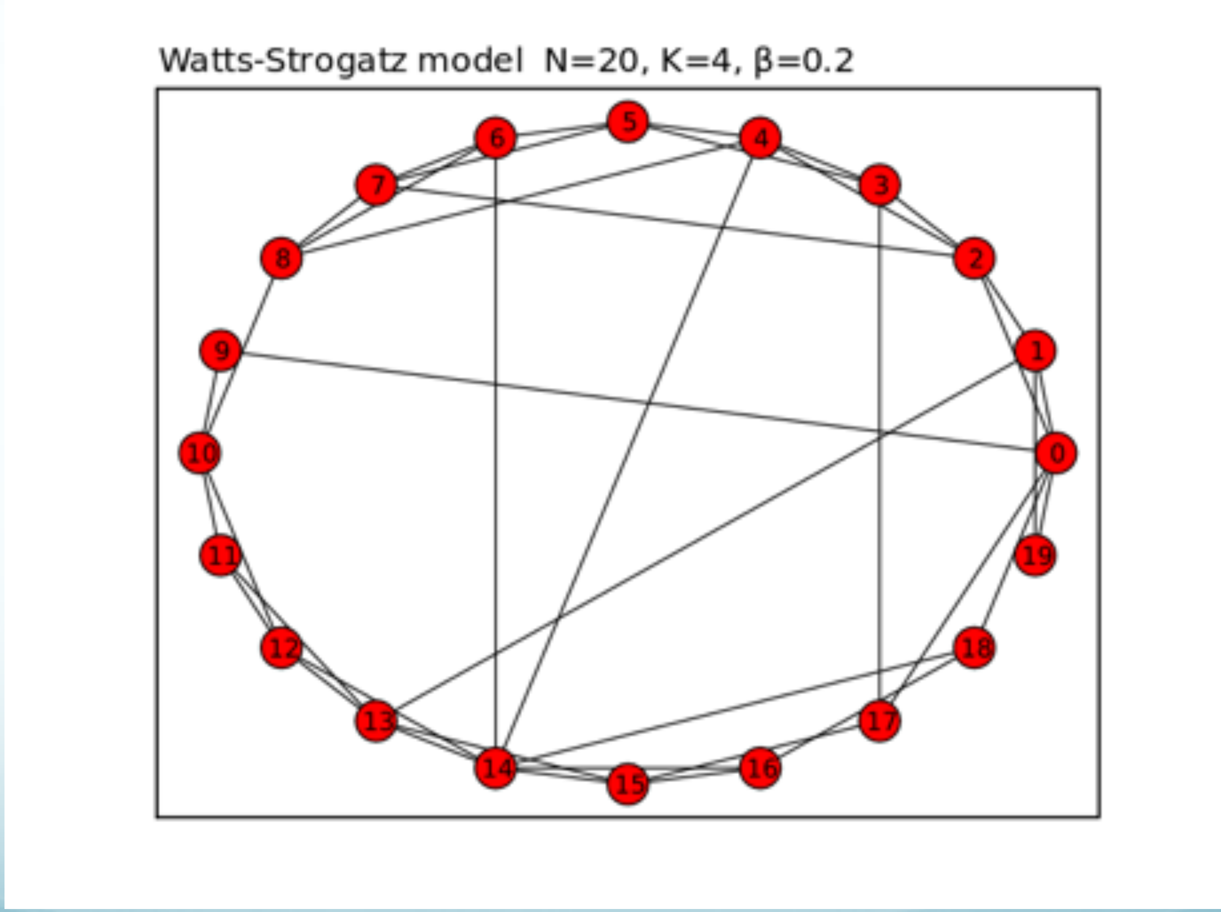
(i) the set of local contacts among all nodes (i.e. the underlying grid structure)

(ii) the location, on the lattice, of the target (*t*)

De-centralized Algorithm

- $\mathcal{A}$: in each step, the current message-holder (*u*) chooses a contact that is as close to the target (*t*) as possible, in the sense of lattice distance.

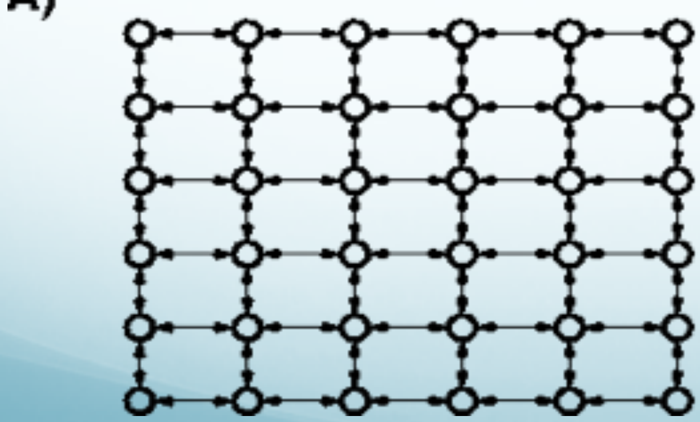
Watts & Strogatz Model



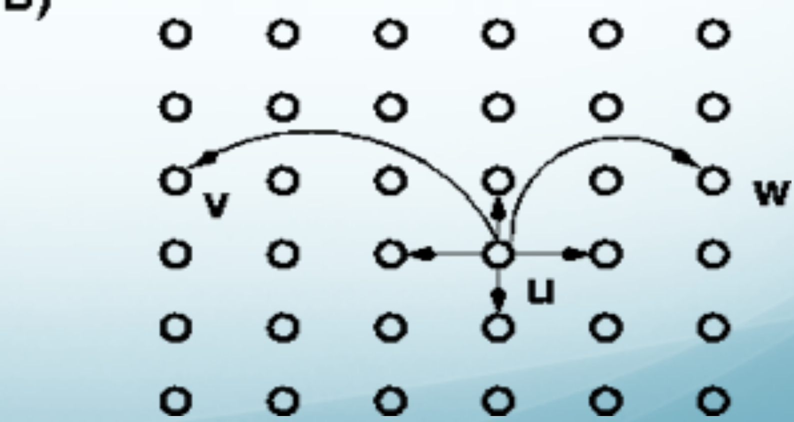
Kleinberg Model

- G(n,p,q,r)
- Distance between verticeies: $d((i,j),(k,\ell)) = |k-i| + |\ell-j|.$
- Probability of long links: $[d(u,v)]^{-r}.$
- Normalizing Factor: $\sum_v [d(u,v)]^{-r}$
- Inverse r-th power distribution

G(6,1,0,0)



G(6,0,2,0)



Kleinburg's Theorems

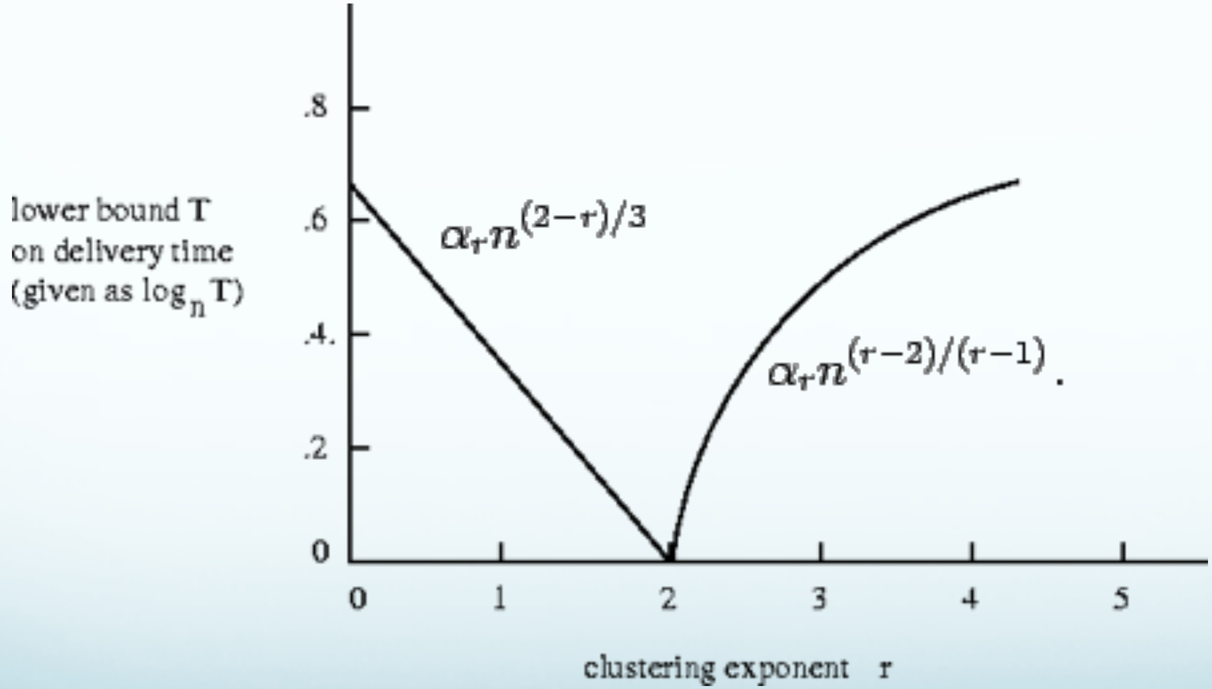
- **Theorem 1** There is a constant α_0, depending on *p* and *q,* but independent of *n*, so that when *r* = 0, the expected delivery time of any decentralized algorithm is at least $\alpha_0 n^{2/3}$ (Hence exponential in the expected minimum path length.)

- **Theorem 2** There is a decentralized algorithm $\mathcal{A}$ and a constant α_2, independent of *n*, so that when *r* = 2 and *p* = *q* = 1, the expected delivery time of is at most $\alpha_2 (\log n)^2$.

- **Theorem 3** (a) Let $0 \le r < 2$. There is a constant α_r, depending on *p*, *q*, *r*, but independent of *n*, so that the expected delivery time of any decentralized algorithm is at least $\alpha_r n^{(2-r)/3}$.

- (b) Let *r* > 2. There is a constant α_r, depending on *p*, *q*, *r*, but independent of *n*, so that the expected delivery time of any decentralized algorithm is at least $\alpha_r n^{(r-2)/(r-1)}$.

$r = 2$ is the only value for which there is a decentralized algorithm capable of producing chains whose length is a polynomial in (log n)



K-Dimensions

- r=k
- k-th inverse distribution for (log n) in polynomial time