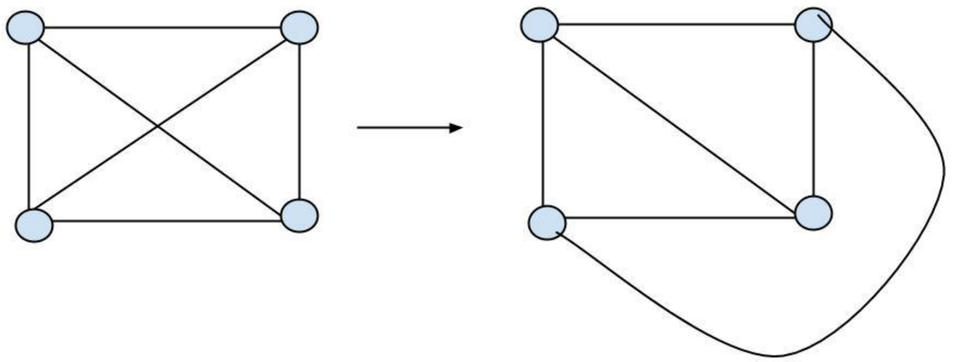
Theorem. (Thomassen [1994b]) Planar graphs are 5-choosable.

Hoa Tang

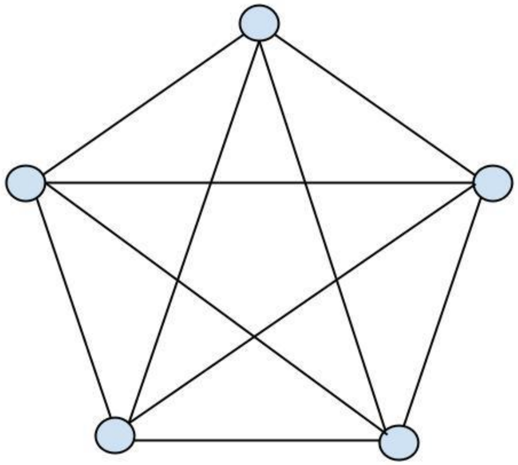
What is Planar Graph?

- A planar graph is a graph that can be drawn in the plane without edge-crossing.

Planar graph

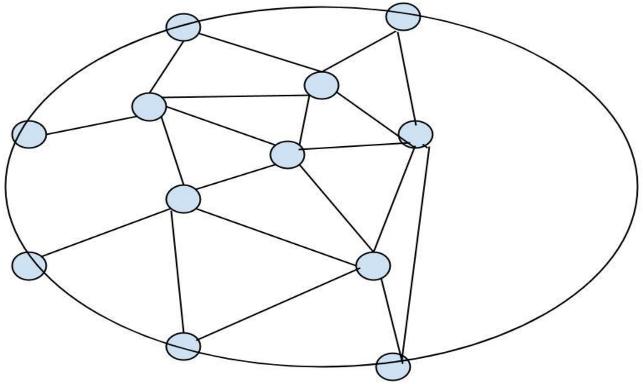


Non-Planar Graph



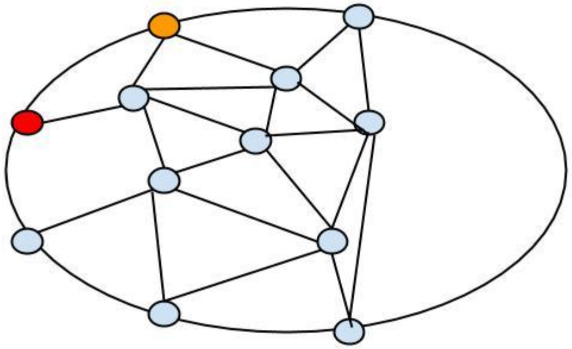
Theorem. (Thomassen [1994b]) Planar graph are 5-choosable.

•**Proof**: Adding an edge never change the list-chromatic number, so we may restrict our attention to plane graphs in which the outer face is a cycle and every bounded face is a triangle.



•**<u>Induction Hypothesis:</u>**

We prove stronger result than the theorem.



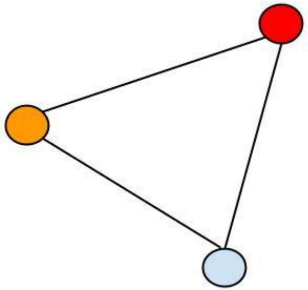
Graph G with k vertices in which 2 vertices on the external cycle are colored.

We defined a stronger list than the 5 list :

- Vertices on the external cycle has list of color with size >= 3
- Vertices in the inside has list of color with size >=5.

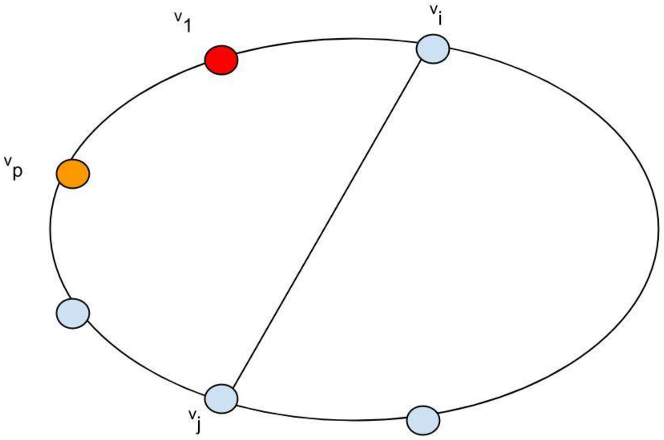
Then graph G is choosable with the given list

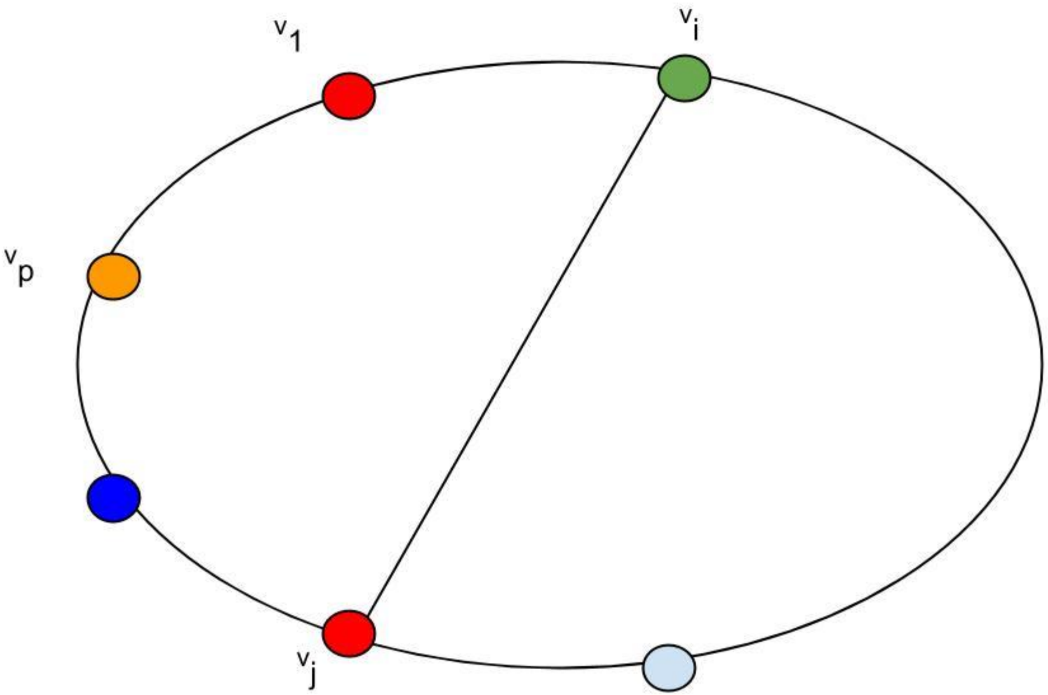
- **<u>Base case:</u>** n = 3, a color available for the third vertex.



•**Induction Step:** Consider n > 3. Let v_p, v_1 be the vertices with fixed colors on the external cycle C. Let $v_1,\ldots, v_p$ be V(C) in clockwise order. We have 2 cases:

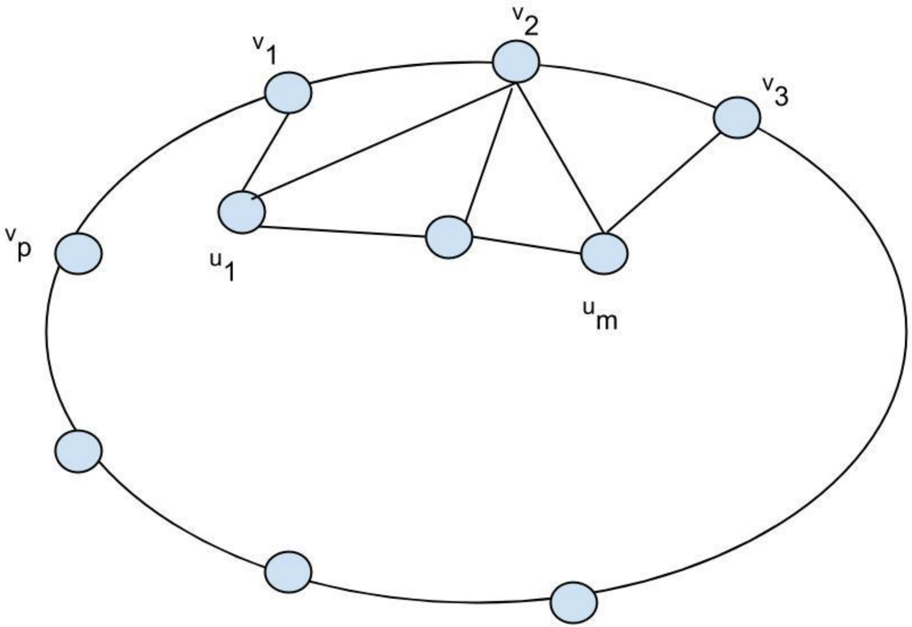
•**Case 1:** C has a chord $v_i\, v_j$ with $1 \le l \le j - 2 \le p - 2$.





v1
vi
vp
vj

•**Case 2:** C has no chord



8.4.32. Theorem. (Thomassen [1994b]) Planar graphs are 5-choosable.

Proof: Adding edges cannot reduce the list chromatic number, so we may restrict our attention to plane graphs where the outer face is a cycle and every bounded face is a triangle. By induction on $n(G)$, we prove the stronger result that a coloring can be chosen even when two adjacent external vertices have distinct lists of size 1 and the other external vertices have lists of size 3. For the basis step ($n = 3$), a color remains available for the third vertex.

Now consider $n > 3$. Let v_p, v_1 be the vertices with fixed colors on the external cycle C. Let $v_1, \ldots, v_p$ be $V(C)$ in clockwise order.

Case 1: C has a chord v_iv_j with $1 \le i \le j - 2 \le p - 2$. We apply the induction hypothesis to the graph consisting of the cycle $v_1, \ldots, v_i, v_j, \ldots, v_p$ and its interior. This selects a proper coloring in which v_i, v_j receive some fixed colors. Next we apply the induction hypothesis to the graph consisting of the cycle $v_i, v_{i+1}, \ldots, v_j$ and its interior to complete the list coloring of G.

Case 2: C has no chord. Let $v_1, u_1, \ldots, u_m, v_3$ be the neighbors of v_2 in order ($3 = p$ is possible). Because bounded faces are triangles, G contains the path P with vertices $v_1, u_1, \ldots, u_m, v_3$. Since C is chordless, $u_1, \ldots, u_m$ are internal vertices, and the outer face of $G' = G - v_2$ is bounded by a cycle C' in which P replaces v_1, v_2, v_3.

Let c be the color assigned to v_1. Since $|L(v_2)| \ge 3$, we may choose distinct colors $x, y \in L(v_2) - \{c\}$. We reserve x, y for possible use on v_2 by forbidding x, y from $u_1, \ldots, u_m$. Since $|L(u_i)| \ge 5$, we have $|L(u_i) - \{x, y\}| \ge 3$. Hence we can apply the induction hypothesis to G', with $u_1, \ldots, u_m$ having lists of size at least 3 and other vertices having the same lists as in G. In the resulting coloring, v_1 and $u_1, \ldots, u_m$ have colors outside $\{x, y\}$. We extend this coloring to G by choosing for v_2 a color in $\{x, y\}$ that does not appear on v_3 in the coloring of G'. ■