Gallai-Roy-Vitaver Theorem

Theorem: If D is an orientation of a graph G with longest path length L(D), then X(G) ≤ L(D) + 1. Furthermore, equality holds for some orientation of G

Proof of First Statement of Gallai-Roy-Vitaver Theorem

- Given an orientation D of a graph G, let D' be a maximal acyclic subdigraph.
- Colour all *v* in V(G) by letting *f(v)* = 1 plus the longest path in D' ending at *v*

Proof of First Statement of Gallai-Roy-Vitaver Theorem

- Let P be a path in D', and *u* the first vertex of P.
- Since D' is acyclic, no path ending at *u* has another point on P.
- Therefore any path ending at *u* (including the longest such path) can be lengthened along P.
- This implies that *f* strictly increases along each path in D'.

Proof of First Statement of Gallai-Roy-Vitaver Theorem

- The colouring *f* uses colours 1 through 1 + L(D') on V(D') (which is equal to V(G))
- For every edge (*u,v*) in E(D) there is a path between its endpoints in D', since (*u,v)* is in E(D') or its addition creates a cycle.
- This implies $f(u) \neq f(v)$ since *f* strictly increases on every path of D'
- So *f* is a proper colouring

Proof of Second Statement of Gallai-Roy-Vitaver Theorem

- To prove this statement we construct an orientation D* such that $L(D^*) \leq X(G) - 1$.
- Let *f* be an optimal colouring of G.
- For each edge *(u,v)* in G, orient it from *u* to *v* if and only if $f(u) < f(v)$.
- Since *f* is a proper colouring, D* is an orientation of G.
- Since the values of *f* increase along each path, and we have X(G) values of *f*, we have $L(D^*) \leq X(G) - 1$

Proof of Second Statement of Gallai-Roy-Vitaver Theorem

- Therefore, we have shown that, given an orientation D of G, $X(G) \leq L(D) + 1$
- We have also shown that there exists an orientation D* such that $L(D^*) \leq X(G) - 1$
- So we see that for D*, $L(D^*) \leq X(G) - 1 \leq L(D^*)$