Kronecker Graph Model

Motivation

- How do we model real networks?

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- Real networks exhibit surprising properties:
  - Heavy tails for in and out degree distributions
  - Small Diameters
  - Densification and shrinking diameter over time

Main Idea

- Generate self-similar graphs recursively
- Begin with an *initiator* graph K_1 with N_1 vertices and E_1 edges
- Recursively generate successively larger self-similar graphs K_2, K_3, … such that the k^{th} graph K_k has $N_k = N_1^k$ vertices
- To do this we can use the Kronecker Product of two matrices

Kronecker Product of Matrices

- Given two matrices **A** and **B** of sizes *m x n* and *p x q* respectively,
- The Kronecker product A ⊗ B is the *mp* × *nq* block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

(https://en.wikipedia.org/wiki/Kronecker_product)

Kronecker Product of Matrices

- More explicitly, this gives:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{bmatrix}.$$

(https://en.wikipedia.org/wiki/Kronecker_product)

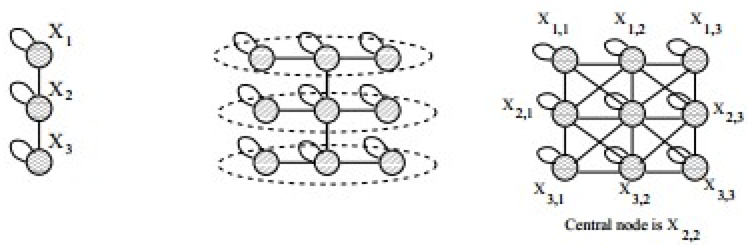
Kronecker Product of Graphs

- Given graphs G and H with adjacency matrices A(G) and A(H) respectively, the Kronecker Product $G \otimes H$ is defined as the graph with adjacency matrix $A(G) \otimes A(H)$

Observation on Edges

- The edge $(X_{ij},X_{kl}) \in G \otimes H$ iff $(X_i,X_j) \in G$ and $(X_k,X_l) \in H$
- This follows from the definition of the Kronecker Product

Example



(a) Graph K_1 (b) Intermediate stage (c) Graph $K_2 = K_1 \otimes K_1$

1	1	0
1	1	1
0	1	1

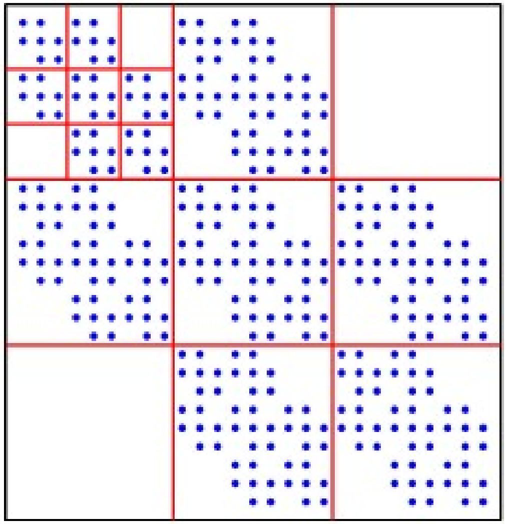
(d) Adjacency matrix of K_1

K_1	K_1	0
K_1	K_1	K_1
0	K_1	K_1

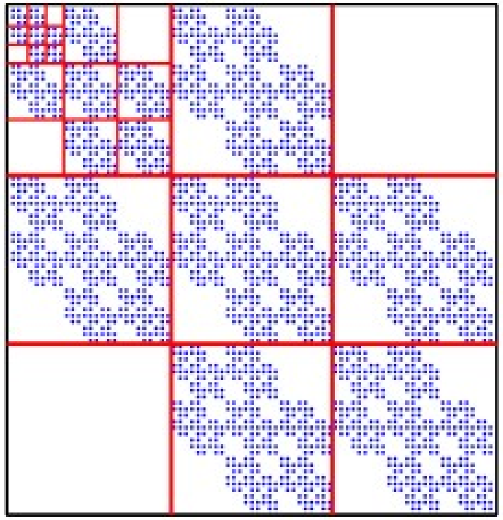
(e) Adjacency matrix of $K_2 = K_1 \otimes K_1$

Example of Kronecker multiplication: Top: a "3-chain" initiator graph and its Kronecker product with itself. Each of the X_i nodes gets expanded into 3 nodes, which are then linked using Observation 1. Bottom row: the corresponding adjacency matrices. See Figure 2 for adjacency matrices of K_3 and K_4.

Example 2



(a) K_3 adjacency matrix (27×27)



(b) K_4 adjacency matrix (81×81)

Definitions

- The k^{th} power of K_1 is defined as the matrix $K_1^{[k]}$, (abbreviated K_k) such that:

$$K_1^{[k]} = K_k = K_1 \otimes K_1 \otimes K_1 \otimes \ldots \otimes K_1 = K_{k-1} \otimes K_k$$

- Kronecker Product of order k is defined by the adjacency matrix $K_1^{[k]}$, where K_1 is the Kronecker initiator adjacency matrix

- The self-similarity of Kronecker graphs is evident in the examples.
- To produce K_k from K_{k-1} , we “expand” each vertex in K_{k-1} by converting it into a copy of K_1 and joining the copies according to the adjacencies in K_{k-1}

- We can imagine this process as positing that communities within the graph grow recursively, with vertices in the community recursively getting expanded into miniature copies of the community. Vertices in the sub-community then link among themselves and also to vertices from other communities.

- The self similarity of Kronecker Graphs can be seen in the examples
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