Connectivity of k-critical graphs

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Math 4330: Topics in Graph Theory
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Lemma (Kainen)

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Proof. Let $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ be the partitions of X and Y formed by the colour classes in proper k-colourings of $G[X]$ and $G[Y]$. If there is no edge between X_i and Y_j then $X_i \bigcup Y_j$ is an independent set in G.

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Form a bipartite graph H with vertices $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$, putting $X_i Y_j \in E(H)$ if in G there is no edge between the set X_i and the set Y_j. If $|[X, Y]| < k$, then H has more than $k(k-1)$ edges.

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In G, we give colour i to all of X_i and all of the set Y_j to which it is matched by M. Since there are no edges joining X_i and Y_j, doing this for all i produces a proper k-colouring of G, which contradicts the hypothesis that $\chi(G) > k$. Hence $|[X, Y]| \geq k$. □

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