Definition: The **cartesian product** of G and H, written $G \square H$ is the graph with vertex set $V(G) \times V(H)$ specified by putting (u,v) adjacent to (u',v') if and only if

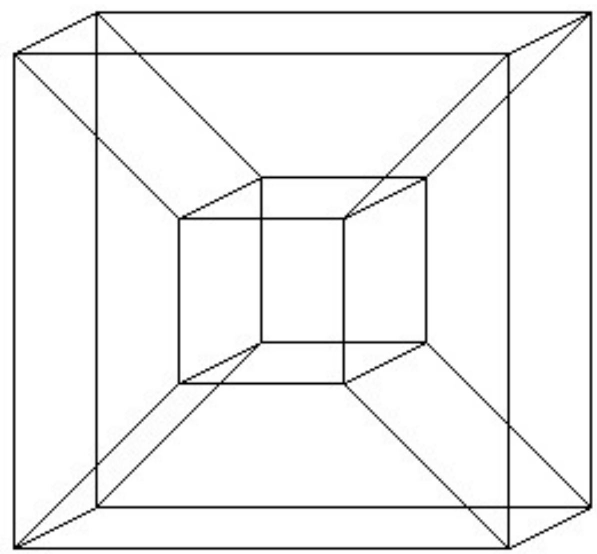
(1) $u = u'$ and $vv' \in E(H)$,

or

(2) $v = v'$ and $uu' \in E(G)$.

Example: $C_3 \square C_4$
The cartesian product operation is symmetric;
$G \square H \cong H \square G$.

The hypercube is another familiar example.



Proposition: $\chi(G\square H)=\max\{\chi(G),\chi(H)\}$

Proof: The cartesian product $G\square H$ contains copies of G and H as subgraphs, so $\chi(G\square H)\geq\max\{\chi(G),\chi(H)\}$

Let $k=\max\{\chi(G),\chi(H)\}$. To prove the upper bound, we produce a proper k-colouring of $G\square H$ using optimal colourings of G and H. Let g be a proper $\chi(G)$-colouring of G, and let h be a proper $\chi(H)$-colouring of H.

Define a colouring f of $G\square H$ by letting $f(u,v)$ be the congruence class of $g(u)+h(v)$ modulo k. Thus f assigns colours to $V(G\square H)$ from a set of size k.

We claim that f properly colours $G\square H$. If (u,v) and (u',v') are adjacent in $G\square H$,then $g(u)+h(v)$ and $g(u')+h(v')$ agree in one summand and differ by between 1 and k in the other. Since the difference of the two sums is between 1 and k, they lie in different congruence classes modulo k.

The cartesian product allows us to compute chromatic numbers by computing independence numbers, because a graph G is m-colourable if and only if the cartesian product $G \square K_m$ has an independent set of size $n(G)$.