Turán graphs

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First, a definition…

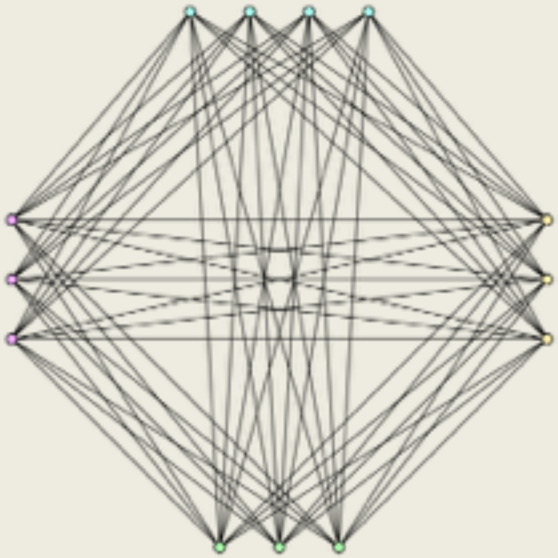
Definition: A **complete multipartite graph** is a simple graph G whose vertices can be partitioned into sets so that u <-> v if and only if u and v belong to different sets of the partition. Equivalently, every component of $\overline{G}$ is a complete graph.

A complete k-partite graph is k-chromatic where the partite sets are the colour classes in the only proper k-colouring.

An example of one of these graphs is a **Turán graph**

A **Turán graph** $T_{n,r}$ is the complete *r*-partite graph with n vertices whose partite sets differ in size by at most 1. By the pigeonhole principle, some partite set has at least size $\lceil n/r \rceil$ and some has size $\lfloor n/r \rfloor$. Therefore, differing by at most 1 means that they all have size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Let $a = \lfloor n/r \rfloor$. After putting *a* vertices in each partite set, $b = n-ra$ remain, so $T_{n,r}$ has b partite sets of size $a+1$ and $r-b$ partite sets of size *a*.



$T_{13,4}$
(Wikipedia)

Lemma: Among simple *r*-partite (*r*-colourable) graphs with n vertices, the Turán graph is the unique graph with the most edges.

Theorem: (Turán 1941) Among the *n*-vertex simple graphs with no *r*+1 clique, $T_{n,r}$ has the maximum number of edges.

Proof: The Turán graph $T_{n,r}$, like every *r*-colourable graph, has no *r*+1 clique, since each partite set contributes at most one vertex to each clique. If we can prove the maximum is achieved by an *r*-partite graph, then the previous Lemma implies that the maximum is achieved by $T_{n,r}$. Thus it suffices to prove that if G has no *r*+1-clique, then there is an *r*-partite graph H with the same vertex set as G and at least as many edges.

We prove this by induction on *r*…

Induction base: $r = 1$.

G and H have no edges.

Induction step: $r > 1$.

Let G be an n-vertex graph with no r+1-clique, and let $x \in V(G)$ be a vertex of degree $k = \Delta(G)$. Let G' be the subgraph of G induced by the neighbours of x. Since x is adjacent to every vertex in G' and G has no r+1-clique, the graph G' has no r-clique. We can thus apply the induction hypothesis to G'; this yields an $r-1$-partite graph H' with vertex set $N(x)$ such that $e(H') \geq e(G')$.

Let H be the graph formed from H' by joining all of $N(x)$ to all of $S = V(G) - N(x)$. Since S is an independent set, H is r-partite. We claim that $e(H) \geq e(G)$. By construction, $e(H) = e(H') + k(n-k)$. We also have $e(G) \leq e(G') + \Sigma_{v \in S} d_G(v)$, since the sum counts each edge of G once for each endpoint it has outside $V(G')$. Since $\Delta(G) = k$, we have $d_G(v) \leq k$ for each $v \in S$, and $|S| = n - k$, so $\Sigma_{v \in S} d_G(v) \leq k(n-k)$. As desired, we have

$$e(G) \leq e(G') + k(n-k) \leq e(H') + k(n-k) = \leq e(H)$$