Calum MacRury B00603090 Jan. 14/15

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Problem Set 1

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1. Let $0 < r \leq n$. $A_1, \ldots, A_n$ are sets for which $|A_i| \geq r$ $1 \leq i \leq n$

Suppose HC holds

then there are at least $r!$ SDRs.

Pf:

Induction on n

Base case: $n = 1$

If $n = 1$, then $r = 1$ as $0 < r \leq 1$

so $|A_1| \geq 1$, and so any member of A_1 will be an SDR, Thus we know there are $1 = r!$ SDR's. ✓

Now assume strong induction:

For all K $1 \leq K < n$, if $A_1, \ldots, A_K$ are sets and r is s.t $0 < r \leq K$, provided $|A_i| \geq r$ for $i = 1, \ldots K$, if HC holds then there are at least $r!$ SDRS. ✓

Now consider $A_1, \dots, A_n$ for which for a given r, $0 < r \le n$, we know $|A_i| \ge r$ for $i = 1 \dots n$.

Assume that HC holds.

Case 1:

Suppose for each $J \overset{\text{strict}}{\subset} \{1, \dots, n\} = [n]$,

$$|A(J)| > |J|$$

Now there are at least r elements to pick from A_n as $|A_n| \ge r$

Suppose we choose x.

Define $A_i' = A_i \setminus \{x\}$ $i = 1 \dots n-1$

Consider the new collection $A_1' \dots A_{n-1}'$

We know that $|A_i'| \ge r-1$ for $i = 1 \dots n$ ✓

and that $r-1 \le n-1$

We might as well assume $r-1 > 0$, for if $r-1 = 0$, we need only find one SDR and this follows from H

Now HC holds for $A_1' \ldots A_{n-1}'$

as we know that for $J \subseteq [n-1]$,

$|A(J)| > |J|$

But since $A'(J) \geq A(J) - 1$,

we have $A'(J) \geq |J|$

Thus our induction hypothesis may be applied to $A_1', \ldots, A_{n-1}'$, giving us at least $(r-1)!$ SDR's from this collection. ✓

We may add on x to build at least $(r-1)!$ SDR's from $A_1, \ldots, A_n$, as $x \notin \bigcup_{j=1}^{n-1} A_j'$. ✓

There were at least r choices for x, and so at least $r!$ SDR's to build is a member of $\bigcup_{j \in J} A_j$

Thus
$$|A(J \sqcup J_0)| = |A(J)| + |A'(J_0)|$$
$$= |J| + |A'(J_0)|$$
$$< |J| + |J_0|$$

Yet this is a contradiction as

HC holds for $\{1, \dots, n\}$ and $J \sqcup J_0 \subseteq [n]$

Thus there is an SDR

from those A'_j for which $j \notin J$

Since $\bigcup_{i \in [n] \setminus J} A'_i \cap \bigcup_{j \in J} A_j = \emptyset$

we may use this SDR to extend

each of our $r!$ SDRs from before

(Over the subcollection $A_1, \dots, A_{m=|J|}$)

to SDRs over the collection

$A_1, \dots, A_n$ ✓

Case 2:

Suppose $\exists J \subset [n]$ s.t

$$|A(J)| = |J|$$

W.l.o.g assume $A_1, \dots, A_m$ are the selected sets where $1 \leq m = |J| < n$ strict

(Otherwise we may relabel our sets this way) ✓

Now consider A_1

We know $r \leq |A_1| \leq |A(J)| = |J| = m$

as $A_1 \subseteq \bigcup_{i=1}^{m} A_i$

Moreover HC holds for $A_1, \dots, A_n$ so it holds for our subcollection $A_1 \dots A_m$

Yet we also know that $m < n$, thus we may apply strong induction yielding at least $r!$ SDR's.

built from our subcollection

Perfect!

2. a) $A_1, \ldots, A_n$ is a collection of sets where $|A_i| \geq r$ for $i = 1 \ldots n$ and each element of $\bigcup_{i=1}^{n} A_i$ occurs in at most r sets of the collection.

I will show that HC holds for the collection.

Pf:

Suppose $q = |\bigcup_{i=1}^{n} A_i|$ and $x_1, \ldots, x_q$ are the members of $\bigcup_{i=1}^{n} A_i$

Define matrix M by

$$(m_{ij}) = \begin{cases} 1 & \text{if } x_i \in A_j \\ 0 & \text{o/w} \end{cases}$$

for $i = 1 \ldots q$ and $j = 1 \ldots n$

Now consider $J \subseteq \{1, \ldots, n\}$

We wish to show $|J| \leq |A(J)|$

Consider the sum s defined as

$$s = \sum_{j \in J} \sum_{\substack{i \in \{1, \dots q\} \\ X_i \in A_j}} m_{ij}$$

Since for each $j \in J$ we know that $|A_j| \geq r$, at least r entries in the matrix's j^{th} column will be 1's

Thus $s \geq |J| r$ ✓

On the other hand, we can reverse the order that s is summed by noticing

$$s = \sum_{j \in J} \sum_{\substack{i \in \{1, \dots, q\} \\ X_i \in A_j}} m_{ij} = \sum_{\substack{i \in \{1, \dots, q\} \\ X_i \in A(J)}} \sum_{\substack{j \in J \\ m_{ij} = 1}} m_{ij}$$

Now for each $X_i \in A(J)$, we know X_i occurs in at most r sets of our subcollection, and so for the i^{th} row of M, we will find at most r columns for which $m_{ij} = 1$ $j \in J$. Thus $s \leq |A(J)| r$ ✓.

Assuming $r > 0$, we may deduce

that $|J| r \leq s \leq |A(J)| r$

so $|J| \leq |A(J)|$ //

2 b.)

For this problem, there is no winning strategy for Bob.

Let $A_1, \ldots, A_{13}$ be the 13 stacks each containing 4 cards.

Since cards with the same suit are considered identical, there are 13 types of cards. (So $A_j \subseteq \{1, \ldots, 13\}$ $j = 1 \ldots 13$)

Define M a matrix where for

$i = 1 \ldots 13$ $\quad j = 1 \ldots 13$

$$(m_{ij}) := \left\{ \begin{array}{l} \text{the number of times} \\ \text{card of type } i \\ \text{occurs in stack } A_j \end{array} \right.$$

To be clear, the following correspondence exists between cards and their type

Ace - 1 Jack - 11 Queen - 12 K

So while a stack A_j may have size $1 \le |A_j| \le 4$, depending on how many copies of each type of card it has, each stack has 4 cards.

Suppose $J \subseteq \{1, \dots, 13\}$

Then $|A(J)| \ge |J|$

This is because if

$$s := \sum_{j \in J} \sum_{i \in A_j} m_{ij}$$

We notice each A_j contains 4 cards, and so adding the j^{th} column yields 4.

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Thus $s = 4 \cdot |J|$

On the other hand, adding row by row, there are $|A(J)|$ rows to add, each with sum at most 4, depending on how many times a card of type $i \in A(J)$ is found among our chosen stacks.

Thus $4 \cdot |J| = s \le 4|A(J)|$, so HC holds and we have an SDR

3. Consider

$$P_1 \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \\ 2 & 1 & 4 & 3 \\ \{3,4\} & \{3,4\} & \{1,2\} & \{1,2\} \end{array}$$

and

$$P_2 \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \\ 4 & 1 & 2 & 3 \\ \{2,3\} & \{3,4\} & \{4,1\} & \{2,1\} \end{array}$$

P_1 has $|\{3,4\}| \times |\{2,1\}| = 4$ possible completions. This is because the sets $\{1,2\}$ and $\{3,4\}$ are disjoint and so we may complete the first two colums (2 possibilities) and then the second two colums (2 possibilities) independently.

6. Given a collection of subsets $A_1, \ldots, A_n$. If, for all $J \subseteq \{1, \ldots, n\}$ $|A(J)| \geq |J| - r$, then there is a subcollection of $n-r$ of these sets which have an SDR

Pf: Let $A_1, \ldots, A_n$ be a collection of sets. Let B be a set for which $B \cap A_i = \emptyset$ for $i = 1 \ldots n$ and for which $|B| = r < n$.

Define $A_i' = A_i \cup B$ for $i = 1 \ldots n$

Then we know $|A_i'| = |A_i| + |B|$ as $A_i \cap B = \emptyset$ for $i = 1 \ldots n$

Provided we assume that for all $J \subseteq \{1, \ldots, n\}$,

$$|A(J)| \geq |J| - r$$

we notice that this implies

$$|A'(J)| \geq |J| \quad \text{for all } J \subseteq [n]$$

But this is just HC for the collection $A_1', \ldots, A_n'$

Thus there is an SDR for this collection.

Suppose $x_1, \ldots, x_n$ is this SDR where $x_i \in A_i'$ for $i = 1 \ldots n$

Since $|B| = r$ and the x_i elements are distinct, at most r of these elements are from B

If we look at the ones not from B, we have an SDR on a subcollection of $A_1, \ldots, A_n$. This subcollection is at least size $n-r$. //

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