Integer Programming and Constraint Programming applied to the Mutually Orthogonal Latin Squares Problem

by

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1 Introduction

Existence of three 10×10 mutually orthogonal latin squares ($MOLS$) is a famous unsolved problem in the theory of latin squares. This paper describes the results of our attempt to solve it by using 0-1 integer programming (IP) and constraint programming (CP).

Before attacking the 10×10 triple problem we decided to develop IP and CP tools to prove the well known result that there cannot exist a pair of 6×6 $MOLS$. There are at least three possible IP models for finding a pair of $MOLS$, one with $O(n^2)$ variables ranging over n values each, another with $O(n^3)$ 0-1 variables and $O(n^3)$ constraints, and the one we adopted which has n^4 0-1 variables and $6n^2$ equality constraints. The latter formulation is also referred to in Dantzig's textbook [Dantzig1963], which quotes David Gale as suggesting that IP be tried in this area.

Below we provide a partial description of the facial structure of the integer polytope for the problem of finding a pair of $MOLS$ - the OLS polytope. This work builds on the previous work of [Balas1989],[Balas1990] and [Euler1985],[Euler1986] on the 3-index axial and planar assignment problems - $3AP_n$ and $3PAP_n$ - respectively. We show that our problem is equivalent to $4PAP_n$ and that the dimension of the OLS polytope can be derived by carefully generalising the relevant results for the $3AP_n$ and $3PAP_n$ polytopes. Starting with the intersection graph of the problem, we are able to derive three types of cliques for the OLS polytope, which are shown to be mutually exclusive and collectively exhaustive. Two types of cliques are shown to induce facet-defining inequalities, while the third gives rise to the minimum equality system defining the poytope. Next, we provide $O(n^4)$, i.e. linear in the number of variables, separation algorithms for the two types of facet-defining clique inequalities. We also develop several types of odd-hole inequalities, for some of which efficient lifting procedures and separation algorithms have been devised. Using some of these valid inequalities in a branch and cut environment we are able to show (eventually) that a pair of 6×6 $MOLS$ cannot exist.

Recently we have been able to extend the formulation used for the pair of MOLS to the problem of finding three 10×10 $MOLS$. This formulation uses 5 indices, has n^5 0-1 variables and $10n^2$ constraints. It is possible to adept this formulation to model the general problem of finding k $MOLS$ of order n, where $k \leq n-1$. We are fairly confident that all the results on the facial structure of the OLS polytope can be generalised to the polytope of the general $MOLS$ problem.

Computationally, the best results are achieved by applying CP. Starting with a 2-index formulation with $O(n^2)$ variables and using the 'all_different' operator judiciously, the non-existence of a pair of 6×6 $MOLS$ can be proved in less than 6 seconds. Moreover, a pair of 10×10 $MOLS$ can be created in less than an hour. Hence, using modern computers and CP, Euler's famous conjecture about the non-existence of a pair of $MOLS$ for $n = 4k+2$ [Euler1779] can be proved to be wrong in less than an hour by demonstrating the existence of a pair for $k = 2$.

Applications of the $MOLS$ problem, its equivalence to other problems and some lessons in modelling with IP and CP are also reported.

2 MOLS and Related Problems

Problems related to MOLS have been extensively studied in the past. Here we present some basic facts and definitions:

Definition 1 *A latin square L is a square matrix L having n^2 entries of n different elements, each one occurring exactly once in each row and in each column. Without loss of generality, we can assume that the n elements are the integers $0, 1, ..., n-1$. The number n is called the order of the Latin square.*

Definition 2 *Two latin squares $L_1 = ||a_{ij}||, L_2 = ||b_{ij}||$ on n symbols $0, 1, ..., n-1$ are said to be orthogonal, if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}), $i, j = 0, 1, ..., n-1$. This definition is naturally extended to a set of more than two latin squares, which are said to be mutually orthogonal, if and only if they are pairwise orthogonal. The maximum number of $MOLS$ of order n is $n-1$.*

An extensive study of the theory of latin squares can be found in [Dénes1974] and [Dènes1991]. We illustrate with an example of a pair of $MOLS$ of order $n = 4$:

0	1	2	3		0	1	2	3
1	0	3	2		2	3	0	1
2	3	0	1		3	2	1	0
3	2	1	0		1	0	3	2

Consider now three disjoint sets $I, J, K, |I| = |J| = |K| = n$ and the collection of all triplets with one element in each set, with a weight associated with each triplet. The *planar four-index assignment* problem ($4PAP_n$) refers to identifying a minimum-weight collection of n^2 triplets, which form n disjoint subsets of n disjoint triplets. Each such subset forms a partition of the union of the three n-sets. One can similarly define the problem $3PAP_n$, asking for a minimum-weight collection of n^2 pairs, which form n disjoint subsets of n disjoint pairs. The above definition is similar to the one presented in [Euler1986]. The $3PAP_n$ is equivalent to the problem of identifying a minimum-weight single latin square, while $4PAP_n$ is equivalent to identifying a minimum-weight pair of $MOLS$ and $kPAP_n$, where $k \leq n+1$, is equivalent to identifying a minimum-weight set of $k-2$ $MOLS$. A special case is the $5PAP_{10}$, i.e. the problem of identifying three $MOLS$ of order 10, the ultimate goal of this research.

The $MOLS$ problem has recently attracted substantial attention, mainly because of its applications in diverse fields. In term of theorerical interest, the $MOLS$ problem is strongly related to the theory of affine and projective planes and to orthogonal hypercubes and (t, m, s)-nets. Its practical applications include the problem of multi-variable experimental design, the problem of designing optimal error-correcting codes and that of encryption. More recently, $MOLS$ have been used to solve scheduling problems and problems related to conflict-free access to parallel memories. See [Laywine1998] for a more extensive list of applications.

3 The Integer Programming Formulation

Concerning the OLS problem of order n, let the latin squares be L_1 and L_2. For the IP formulation of the problem, we use the 0-1 variables $x_{ijkl}, i \in I, j \in J, k \in K, l \in L$. Note that $x_{ijkl} = 1$ if in row i and column j L_1 contains element k and L_2 contains element l; $x_{ijkl} = 0$, otherwise. The formulation is the following (see also [Dantzig1963]):

$$\sum \{x_{ijkl} : i \in I, j \in J\} = 1, k \in K, l \in L$$
$$\sum \{x_{ijkl} : j \in J, k \in K\} = 1, i \in I, l \in L$$
$$\sum \{x_{ijkl} : i \in I, k \in K\} = 1, j \in J, l \in L$$
$$\sum \{x_{ijkl} : k \in K, l \in L\} = 1, i \in I, j \in J$$
$$\sum \{x_{ijkl} : i \in I, l \in L\} = 1, j \in J, k \in K$$
$$\sum \{x_{ijkl} : j \in J, l \in L\} = 1, i \in I, k \in K$$
$$x_{ijkl} \in \{0, 1\}, i \in I, j \in J, k \in K, l \in L \qquad (1.1)$$

The above formulation has n^4 variables and $6n^2$ constraints and is obtained by summing over all possible combinations of two out of four indices. Summing over all four combinations of three indices would yield the constraints of the *axial* four-index assignment problem ($4AP_n$). In general, the IP formulation of the $kPAP_n$ results from summing over all combinations of $k-2$ out of k indices, whereas the the IP formulation of kAP_n results from summing over all k combinations of $k-1$ out of k indices. It is obvious that the OLS, or equivalently the $4PAP_n$, is a special case of the set-partitioning problem.

Let us now introduce some additional notation, which will be used throughout this text. Let A_n be the constraint matrix of (1.1) and G_{A_n} its intersection graph, as introduced in [Padberg1973]. Let $R_n = (I\times J)\cup(I\times K)\cup(I\times L)\cup(J\times K)\cup(J\times L)\cup(K\times L)$ be the row set of A_n and $C_n = I\times J\times K\times L$ the column set of A_n, where $|R_n| = 6n^2$ and $|S_n| = n^4$. Each variable $c \in C_n$ can also be considered as a subset of C_n, e.g. $c = (i_1, j_1, k_1, l_1)$, therefore all set operations, i.e. conjunction, negation, etc., are valid. The support of a (row or column) vector is defined as the set of its non-zero components. We denote with SR^r the support of a row $r \in R$, i.e. $SR^r = \{c \in C_n : a_{rc} = 1\}$, and with SC^c the support of a column $c \in C_n$, i.e. $SC^c = \{r \in R_n : a_{rc} = 1\}$. It is easy to see that the number of non-zero components in each row is n^2, i.e. $|SR^r| = n^2, r \in R_n$. Similarly, $|SC^c| = 6, c \in C_n$.

Let also $P^n = \left\{x \in R^{n^4} : A_n x = e\right\}, e = (1, ..., 1)^T \in R^{n^4}$, be the polyhedron defined by the LP-relaxation of (1.1). The OLS polytope of order n is $P_I^n = conv\left\{x \in \{0,1\}^{n^4} : x \in P^n\right\}$.

4 The intersection graph

Let $G_{A_n} = (C_n, E_n)$ be the intersection graph, with nodes corresponding to columns of A_n and an edge joining two nodes if and only if there is a row of A_n with a non-zero element in the two corresponding columns.

Remark 1 *For $c, d \in C_n, (c,d) \in E_n$ if and only if $\left|SC^c \cap SC^d\right| \geq 1$ or, equivalently, if $|c \cap d| \geq 2$.*

Proposition 3 *The graph $G_{A_n} = (C_n, E_n)$ is regular of degree $2(3n-1)(n-1)$ and $|E_n| = n^4(3n-1)(n-1)$.*

4.1 The cliques of G_{A_n}

A clique is defined as a maximal complete subgraph. There exist three types of cliques in G_{A_n}.We describe these classes and prove that they are mutually exclusive and collectively exhaustive. In the next section we will also prove that the last of these classes defines the minimum equality system of P_I^n, whereas the other two induce facet-defining inequalities in P_I^n. The approach taken to establish these results follows closely the work presented in

[Balas1989] and [Euler1986]. Our work can be viewed as a generalisation of their work from 3-index to 4-index setting.

Proposition 4 *For each $c \in C_n$,, let $D(c) = \{f \in C_n : |c \cap f| = 3\}$. The node set $Cl(c) = D(c) \cup \{c\}$ induces a clique of size $4n-3$ in G_{A_n}. There are n^4 distinct cliques of this type. These cliques will be called cliques of type I throughout this text.*

Proposition 5 *Let $c, d \in C_n$, with $|c \cap d| = 1$. The node set $Cl(c,d) = \{f \in C_n : |c \cap f| = 2, |d \cap f| = 3\} \cup \{c\}$ defines a 4-clique in G_{A_n}. These cliques will be called cliques of type II. The number of distinct cliques of type II is $n^4(n-1)^3$.*

Proposition 6 *For each $r \in R$, the node set SR^r induces a clique of size n^2 in G_{A_n}. There are $6n^2$ cliques of this type. These cliques are to be called cliques of type III.*

Theorem 7 *The cliques of types I, II & III are the only cliques of G_{A_n}.*

Proof: Let Cl be the node set of a clique of $G_{A_n}$and $c = (i_0, j_0, k_0, l_0) \in Cl$. If $|c \cap f| \geq 3$ $\forall\ f \in Cl$, then Cl is of type I. If not, there will be at least one $d \in Cl, s.t. |c \cap d| = 2$, therefore assume w.l.o.g. that $d = (i_0, j_0, k_1, l_1)$. If every other $f \in Cl$ is of the form $f = (i_0, j_0, k, l), k \in K, l \in L$, then Cl is of type III. If not, it can be proved that Cl contains all the elements of a clique of type II. Since, by definition, such a node set is not extendible, Cl must be of type II.

Corollary 8 *The total number of cliques in G_{A_n} is $n^4[(n-1)^3+1]+6n^2$.*

4.2 Separation algorithms for clique inequalities

It is well known that a linear inequality can be associated with each clique Cl of G_{A_n}.The form of such an inequality is $x(Cl) = \sum\{x_c : c \in Cl\} \leq 1$. Let $x \in P^n, x \notin P_I^n$, i.e. a fractional solution to (1.1) . We present two algorithms, which solve the separation problem for inequalities associated with cliques of type I and II, respectively. The algorithms are based on those presented in [Balas1990]. They provide an inequality violated by x, if such an inequality exists.

Algorithm 2.1 Let $x \in P^n$, $x \notin P_I^n$ and v an integer parameter, with $v \geq 5$.

Step 1: Set $d_c = 0$ for all $c \in C_n$.

Step 2: For all $f \in C_n$, check x_f. If $x_f \geq \frac{1}{vn}$, then set $d_c := d_c + x_f$ for all $c \in Cl(f)$. If $d_c > 1$, stop. The inequality $x(Cl(c))$ is violated. Otherwise, continue.

Step 3: For all $c \in C_n$, if $d_c > \frac{v-4}{v}$, then check whether the inequality $x(Cl(c))$ is violated. If so, stop; otherwise continue.

Theorem 9 *Algorithm 2.1 determines in $O(n^4)$ steps whether a given non-integer $x \in P^n$ violates an inequality induced by a clique of type I.*

In practice, the value $v = 8$ minimises the overall complexity.

The proof of this theorem is based on the following intermediate results.

Lemma 10 *For any $x \in P_n$ and any positive number λ, the number of components of x with value $\geq \lambda$ is $\leq \frac{n^2}{\lambda}$.*

Lemma 11 *For any $x \in P_n$ and any positive number λ, the number of $c \in C_n$, such that $x(Cl(c)) \geq \lambda$, is $\leq \frac{n^2(4n-3)}{\lambda}$.*

Algorithm 2.2 Let $x \in P^n, x \notin P_I^n$.

Step 1: For all $c \in C_n$, if $\frac{1}{4} < x_c < 1$, then go to step 2.

Step 2: For all $f \in C_n$, such that $|c \cap f| = 2$, if $x_f > \frac{1-x_c}{3}$, then go to step 3.

Step 3: For all $d \in C_n$, such that, $|c \cap d| = 1, |f \cap d| = 3$, check whether the inequality $x(Cl(c,d)) \leq 1$ is violated. If so, stop; otherwise continue.

Theorem 12 *Algorithm 2.2 determines in $O(n^4)$ steps whether a given non-integer $x \in P^n$ violates an inequality induced by a clique of type II.*

Corollary 13 *Whether there exists a violated clique inequality can be detected in linear time with respect to the number of variables, i.e. in $O(n^4)$ steps.*

4.3 The odd-holes of G_{A_n}

An odd-hole is defined as a chordless cycle of odd length in G_{A_n}. Since for $c, d \in C_n, (c,d) \in E_n$ if and only if $|c \cap d| = 2$ or $|c \cap d| = 3$, we can classify the edges of the odd-hole as *double links* or *triple links,* respectively. Double links can be considered to belong to a set S, $S = I \times J$, $I \times K$, $I \times L$, $J \times K$, $J \times L$, $K \times L$, whereas triple links can belong to a set T, $T = I \times J \times K$, $I \times J \times L$, $I \times K \times L$, $J \times K \times L$. The odd-holes described in [Balas1989] for $3AP_n$ contain single and double links and are easily extended to OLS $(4PAP_n)$, as follows.

Let H be the node-set of an odd-hole in the intersection graph of $3AP_n$, with $c \subset I \times J \times K, \forall c \in H$. Adding $l = l_0$ to all $c \in H$ results in $\widehat{c} = (c, l_0) \subset I \times J \times K \times L$, therefore giving a node set $\widehat{H}$, which induces an odd-hole in G_{A_n}. The single and double links of H become double and triple links in $\widehat{H}$, respectively. This observation allows us to extend all the results of [Balas1989] from $3AP_n$ to our case of OLS $(4PAP_n)$. Odd-holes in G_{A_n} can be associated with row sets of A_n and vice versa.

Based on [Balas1989] one can devise odd-holes of length $2p+1$ in G_{A_n} with t triple links, where $\max\{0, 2(2p+1) - 3n\} \leq t \leq p-1$ and $p \leq n-1$. However, if the method used in [Balas1989] is applied to the $4PAP_n$, we get the inequality $\max\left\{0, \left\lceil \frac{5}{2}(2p+1) - 3n^2 \right\rceil\right\}$ $\leq t \leq p$ and $p \leq \left\lfloor \frac{6n^2-5}{8} \right\rfloor$. Although we believe that this upper bound is weak, we have found families of odd-holes with $p \geq n$.

The classes of lifted odd-hole inequalities described in [Balas1989] are extended in the same way. The number of such inequalities for the $3AP_n$ are $O(2^{3n})$, whereas in our case they are $O(n * 2^{3n})$. The inequalities lifted from 5-holes are separable in linear time, as in [Balas1990], whereas subclasses of inequalities lifted from larger holes remain also polynomially separable.

5 Description of the integer polytope

5.1 The dimension of the polytope

Lemma 14 *The rank of the system* $A_n x = e$ *is* $6n^2 - 8n + 3$, *therefore* $\dim(P^n) = n^4 - 6n^2 + 8n - 3$.

Proof: One can easily see that the upper half of A_n corresponds to the matrix of n independent $3AP_n$ (one for each value of the l index), therefore the linearly dependent rows can be deleted according to [Balas1989](Lemma 3.1). Each column of the lower half of A_n, is repeated exactly n times. Deleting the $n^3(n-1)$ replicated columns, leaves a matrix identical to the one of $3PAP_n$, where the dependent rows can be deleted according to [Euler1986](section 4). For the remaining rows of A_n, one has to check the linear independence for any set of rows, which includes the rows of one of the first three constraints and the rows of one of the last three constraints(9 sets in total). The result follows.

Theorem 15 *For* $n \succeq 5$ *and* $P_I^n \neq \oslash$, *(i.e.* $n \neq 6$*), suppose that the equality* $ax = a_0$, $a \in R^{n^4}, a_0 \in R$, *is satisfied by all* $x \in P_I^n$. *Then there exist scalars* $v^1_{kl}, v^2_{il}, v^3_{jl}, v^4_{ij}, v^5_{jk}, v^6_{ik}$, $i \in I, j \in J, k \in K, l \in L$, *satisfying:*

$$\begin{aligned} a_{ijkl} &= v^1_{kl} + v^2_{il} + v^3_{jl} + v^4_{ij} + v^5_{jk} + v^6_{ik} \\ a_0 &= \sum \{v^1_{kl} : k \in K, l \in L\} + \sum \{v^2_{il} : i \in I, l \in L\} + \sum \left\{v^3_{jl} : j \in J, l \in L\right\} + \\ &\quad \sum \left\{v^4_{ij} : i \in I, j \in J\right\} + \sum \left\{v^5_{jk} : j \in J, k \in K\right\} + \sum \{v^6_{ik} : i \in I, k \in K\} \end{aligned}$$

for all $(i,j,k,l) \in C_n$.

Proof: By defining $v^1_{kl} = a_{11kl}, v^2_{il} = a_{i11l} - a_{111l}, v^3_{jl} = a_{j11l} - a_{111l}, v^4_{ij} = a_{ij11} - a_{i111} - a_{1j11} + a_{1111}, v^5_{jk} = a_{1jk1} - a_{1j11} - a_{11k1} + a_{1111}$, $v^6_{ik} = a_{i1k1} - a_{i111} - a_{11k1} + a_{1111}$ and replacing the values in the equation above, we get:

$$\begin{aligned} a_{ijkl} &= a_{ij11} + a_{i1k1} + a_{i11l} + a_{1jk1} + a_{j11l} + a_{11kl} \\ &\quad -2 * (a_{i111} + a_{1j11} + a_{11k1} + a_{111l}) + 3a_{1111} \end{aligned}$$

Essentially the scalars become zero for all dependent rows. The proof progresses by extensively examining all the possible cases for a_{ijkl}, using index interchanges, as in [Balas1989]. Simple properties of symmetry in latin square have also to be taken into account.

Corollary 16 *The system $A_n x = e$ is the minimum equality system for P_I^n and* $\dim(P_I^n) = n^4 - 6n^2 + 8n - 3$.

5.2 Facets associated with cliques

Theorem 17 *For $n \succeq 5$ and $P_I^n \neq \oslash$, the inequality $\sum\{x_f : f \in Cl(c)\} \leq 1$ defines a facet of P_I^n for all $c \in C$.*

Proof: Let $P_I^{Cl(c)} = conv\{x \in \{0,1\}^{R^4} : A_n x = e, \sum\{x_f : f \in Cl(c)\} = 1\}$. It is sufficient to show that $P_I^{Cl(c)}$ defines a facet of P_I^n. W.l.o.g. assume that $c = (n,n,n,n)$. Then: $Cl(c) = Cl((n,n,n,n)) = \{(n,n,n,n\}, (1,n,n,n), ..., (n-1,n,n,n),\ (n,1,n,n), ..., (n,n-1,n,n),\ (n,n,1,n), ..., (n,n,n-1,n), (n,n,n,1), ..., (n,n,n,n-1)\}$.

Suppose that the equality $ax = a_0$, $a \in R^{n^4}, a_0 \in R$, is satisfied by all $x \in P_I^{Cl(c)}$. Then we can exhibit scalars $v_{kl}^1, v_{il}^2, v_{jl}^3, v_{ij}^4, v_{jk}^5, v_{ik}^6, \pi$ $(\pi \in R)$, $i \in I, j \in J, k \in K, l \in L$, satisfying:

$$a_{ijkl} = \left\{ \begin{array}{l} v_{kl}^1 + v_{il}^2 + v_{jl}^3 + v_{ij}^4 + v_{jk}^5 + v_{ik}^6,\ (i,j,k,l) \in C_n \backslash Cl((n,n,n,n)) \\ v_{kl}^1 + v_{il}^2 + v_{jl}^3 + v_{ij}^4 + v_{jk}^5 + v_{ik}^6 + \pi,\ (i,j,k,l) \in Cl((n,n,n,n)) \end{array} \right\}$$

The proof progresses in the same way as in theorem 16.

Theorem 18 *For $n \succeq 5$ and $P_I^n \neq \oslash$, the inequality $\sum\{x_f : f \in Cl(c)\} \leq 1$ defines a facet of P_I^n for all $c, d \in C$.*

Proof: As in the previous theorem.

6 The Constraint Programming approach

Constraint programming (CP) has recently attracted strong interest from the combinatorial optimization/operational research community. Its generic paradigm can incorporate non-linear constraints and the derived models are more compact. Although it lacks the global perspective given by LP, it appears promising for solving both feasibility and optimality problems. We have applied CP to the $MOLS$ problem, viewed as a feasibility problem, with considerable success. The devised model uses the all_different operator, which, when applied to a variable list, implies that all the list members must possess pairwise different values. Assuming squares X and Y, the CP model for a pair of OLS is the following:

SETS:

$I = \{0, ..., n-1\}, J = \{0, ..., n-1\}$

VARIABLES:

$X_{ij}, i \in I, j \in J$ $\quad$ $Y_{ij}, i \in I, j \in J$ $\quad$ $Z_{ij} = n * X_{ij} + Y_{ij}, i \in I, j \in J$

DOMAINS:

$D_{X_{ij}} = D_X = \{0, ..., n-1\}$ $\quad$ $D_{Y_{ij}} = D_Y = \{0, ..., n-1\}$

$D_{Z_{ij}} = D_Z = \{0, ..., n^2-1\}$

CONSTRAINTS:

$all\_different\{X_{ij}, j \in J\}, i \in I$ $\quad$ $all\_different\{X_{ij}, i \in I\}, j \in J$

$all\_different\{Y_{ij}, j \in J\}, i \in I$ $\quad$ $all\_different\{Y_{ij}, i \in I\}, j \in J$

$all\_different\{Z_{ij}, i \in I, j \in J\}$

To solve the model, a search tree is formed, where each level corresponds to a variable and each node corresponds to a variable being fixed at a specific value. The algorithms implemented follow the *Forward Checking* (FC) scheme, where, after fixing a variable, the domains of the variables not yet instantiated are filtered to exclude inconsistent values. Apart from the above constraints, additional *implicit* constraints are also taken into account by introducing the notion of *degrees of freedom.* The degree of freedom for a row/column and a value is defined as the number of cells in this row/column that still contain the value in their domains. The constraints are that the degree of freedom must remain non-zero for every value not yet instantiated in a row/column of both squares. Those constraints are derived from the following remark, which is presented here only for the X square.

Remark 2 *Let* $S_{i_0} = \{X_{ij} : i = i_0, X_{ij}\ uninstantiated\}$ *and* $D_{i_0} = \left\{\cup D_{X_{ij}} : X_{ij} \in S_{i_0}\right\}, i_0 \in I$ *and* $S_{j_0} = \{X_{ij} : j = j_0, X_{ij}\ uninstantiated\}$ *and* $D_{j_0} = \left\{\cup D_{X_{ij}} : X_{ij} \in S_{j_0}\right\}$. *A necessary condition for the existence of a feasible solution, at any step of the algorithm, is that* $|S_{i_0}| = |D_{i_0}|, i_0 \in I$ *and* $|S_{j_0}| = |D_{j_0}|, j_0 \in J$.

The results obtained for order $n = 4, ..., 12$ are summarised in the table below:

n	4	5	6	7	8	9	10	11	12
$Time$(sec)	0.0	0.0	5.6	65.6	0.1	45,302.3	3,133.2	47,888.4	8,945.0

Note that for $n = 6$ the problem is known to be infeasible ([Dénes1974],[Laywine1998]), therefore the whole solution space has to be explored before infeasibility is proven. Also note the explosion of the solution time for larger orders, especially for the ones being powers of prime numbers.

An extension of the above model has been used to derive sets of $n-1$ MOLS for orders $n = 4, 5, 7$. The results are summarised below:

n	4	5	7
$Time$(sec)	0.0	0.0	17490.8

7 Conclusion

We have examined the OLS ($4PAP_n$) polytope and have proved that $\dim(P^n) = \dim(P_I^n)$, i.e. the polytope given by the LP-relaxation has the same dimension as the polytope described by the convex hull of the integer solutions. A similar result has also been proved for the polytopes of $3PAP_n$ ([Euler1986]), $2AP_n$ and $3AP_n$[Balas1989]. Note also that the outline of all relevant proofs is quite similar. It remains an open question whether the same result is true for higher orders of both planar and axial assignment problems. Addressing this question would certainly require a more general argument than examining each polytope individually. The dimension of $5PAP_n$ is currently under investigation. Further refinements of the CP approach to solve the problem of finding a 10×10 triple of MOLS is also under consideration.

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