Lower Bounds from Tile Covers for the Channel Assignment Problem*

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Abstract

A method to generate lower bounds for the channel assignment problem is given. The method is based on the reduction of the channel assignment problem to a problem of covering the demand in a cellular network by pre-assigned blocks of cells, called tiles. This tile cover approach is applied to networks with a cosite constraint and two different constraints between cells. A complete family of lower bounds is obtained which include a number of new bounds, and improve or include almost all known clique bounds. When applied to an example from the literature, the new bounds give better results.

1 Introduction

Finding an optimal assignment of communication channels in a cellular network is a difficult combinatorial optimization problem, which has received considerable attention over the last decade, due to the explosive growth of wireless communications and the scarcity of the radio spectrum. The channel assignment problem (CAP) is NP-complete even in a drastically simplified form, and consequently most efforts have gone towards the development of good heuristics. (Recently, integer programming techniques have been used which can lead to exact solutions, see for example [9]). Lower bounds play an important role in the evaluation of any heuristic or approximation algorithm. Moreover, lower bounds can help to identify the structures that form the bottleneck for a particular instance, and this information in its turn can be used to find better assignments.

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A basic model describes a cellular network in terms of the *demand* for channels in each cell, and a set of *separation constraints* which prescribe the minimal separation that must exist between channels assigned to certain cells, in order to avoid interference. The goal of the CAP is to assign channels (represented by integers) to the cells such that each cell receives as many channels as its demand requires, while none of the separation constraints are violated. The objective hereby is to minimize the *span* of the assignment, which is the difference between the highest and the lowest channel assigned. A secondary objective, when a limited span is given, can be to minimize the number of violated interference constraints.

Cellullar networks can be modeled as graphs, where the nodes of the graph represent the cells, and two nodes are adjacent precisely when there exists a (non-zero) separation constraint between them. The demands are given by a *weight vector* indexed by the nodes, and the separation constraints are given by a vector indexed by the nodes and edges. When all separation constraints are 1, the CAP reduces to the problem of finding a colouring of a weighted graph.

The minimal span needed for any assignment will generally be determined by the cells with highest demand. It is reasonable to assume that these cells will often be geographically close, corresponding, for example, to a business district or a city center. Since interference also tends to be highest between cells that are close, these cells will often form a clique in the underlying graph.

Most lower bounds for the CAP are therefore based on cliques. The simplest clique bound, mentioned in [4] but generally considered folklore, is given by assuming all edge constraints and cosite constraints to be equal to the lowest constraint in the clique. A first refinement is obtained in [4], by considering two different constraints. A second refinement, similar to the situation studied here, was considered in [14]. The bounds mentioned were obtained using ad-hoc methods.

In this paper, we study networks where the separation constraint between different cells can take only three values, one of which is reserved for the *co-site constraint*. The co-site constraint is the separation constraint between channels assigned to the same cell, or node. Naturally, any bounds obtained from this approach can also be used in networks with more general constraints by reducing the constraints in any particular set of edges to the lowest constraint in that set.

We describe how lower bounds can be generated from an approach based on reducing the CAP to a covering problem. The crucial step is to show that any channel assignment can be broken down into a small blocks called *tiles*. A *tile cover* is a collection of such tiles, so that the number of tiles covering a node equals the number of channels assigned to that node. The conversion of the CAP to a tile cover problem brings the advantage that tile covers can be easily analyzed, using LP duality and polyhedral methods.

We apply the tile cover approach to configurations which we call *nested cliques*, which are cliques that contain a subset of nodes such that all constraints that involve a node from this set take the larger value. Nested cliques arise naturally from the geographical layout of cellular networks, and the fact that interference levels are generally lower between transmitters that are at greater distance from each other. Hence it will be common to find a cluster of cells with high interference constraints between them, surrounded by an outer shell of cells at greater distance, and thus with weaker interference constraints. Such a situation will form a nested clique in the interference graph.

Using the tile cover approach on nested cliques, we derive a comprehensive family of general "second generation" clique bounds. This family includes all bounds from [4], and improve the bound obtained in [14]. We also show, using an example, how the approach can be used directly to obtain a specific set of lower bounds for any specific set of parameters.

Two different kind of clique bounds exist which cannot be directly derived from our tile cover approach. In [12] and [6] it was shown how the Travelling Salesman Problem and its Linear Program relaxation can be used to derive lower bounds for cliques. This approach is most effective when the co-site constraint is relatively low. In [15] a lower bounding method is described which is based on network flows. However, our tile cover bounds give better results when applied to the example given in this paper.

Since it is NP-hard to find the maximum weight clique in a graph, it will also be hard to find the nested clique that gives the best bound. However, clique enumeration procedures such as the *Cardaghan-Pardalos* algorithm (see [3]) give good performance in practice. The reduction of the CAP to a tile cover problem leads to an easy way of computing the lower bound for any particular clique by way of a linear program. Alternatively, any particular network can be analyzed in advance using our method, and a complete family of easily computable lower bounds can be obtained. Therefore, we expect the computation of the best tile cover clique bound to be feasible and realistic.

In Section 3, we describe the tiles that can occur in a tile cover for nested cliques. A *cost* is associated with each tile, which roughly corresponds to the part of the span taken up by assigning channels to the tile. Our main result, proven in Section 5, states that each channel assignment can be reduced to a tile cover, such that the cost of the cover is no larger than the span of the assignment. This then implies that any lower bound on the cost of a tile cover is a lower bound on the span of a channel assignment.

In Section 4, we develop lower bounds for tile covers, which then directly translate into bounds for the CAP. First we formulate the Integer Program which finds tile covers of minimal cost, and then use its LP relaxation, LP duality and polyhedral methods to obtain lower bounds. We show how this approach generates or generalizes the bounds from [4] and [14]. We also give new general bounds for the case where the lowest edge constraint is 1, and for the case where $\lfloor \frac{k}{u} \rfloor = 2$ and $\lfloor \frac{k}{a} \rfloor = 1$, where k is the co-site constraint, and u and a are the two edge constraints. Moreover, we show how the same method could be used to generate lower bounds for any particular choice of parameters. We demonstrate this approach on an instance of the CAP taken from [15], where our methods give an improvement of 13% over the previously best bound.

2 Preliminaries

For the basic definitions of graph theory we refer to [2]. A (simple) graph G is a pair (V, E) of a node set V and an edge set E, where E is a set of 2-subsets of V. A **clique** in a graph is a set of nodes of which every pair is adjacent.

In this paper, we will use the following notation for integer vectors: if $y \in \mathbb{Z}^V$ for some set V, then $y(v)$ is the coordinate of y indexed by v. Sets will often be represented by their characteristic vectors. Given a set V and $A \subseteq V$, the characteristic vector $\chi^A \in \mathbb{Z}_+^V$ is defined as follows:

$$\chi^A(v) = \begin{cases} 1 & \text{if } v \in A \\ 0 & \text{otherwise} \end{cases}$$

Conversely, given a vector $y \in \mathbb{Z}_+^V$, the **support** of y, denoted by $V(y)$, is the set of all nodes in V indexing non-zero coefficients of y, so

$$V(y) = \{v \in V : y(v) > 0\}.$$

A **constrained graph** $G = (V, E, s, e)$ is a graph $G = (V, E)$ and positive integer vectors $s \in \mathbb{Z}_+^V$ and $e \in \mathbb{Z}_+^E$ representing the reuse constraints: the vector s represents the **co-site constraints**, the required separation between channels assigned to the same node, and e represents the **edge constraints**, the required separation between channels assigned to the two endpoints of an edge.

A **constrained, weighted graph** is a pair (G, w) where G is a constrained graph and w is a positive integral weight vector indexed by the nodes of G. The component of w corresponding to node u is denoted by $w(u)$ and called the **weight** of node u. The weight of node u represents the number of channels needed at node u.

A **channel assignment** for a constrained, weighted graph (G, w) where $G = (V, E, s, e)$ is an assignment f of sets of non-negative integers (which will represent the channels) to the nodes of G which satisfies the conditions:

$$\begin{array}{ll} |f(u)| = w(u) & (u \in V), \\ i \in f(u) \text{ and } j \in f(v) \Rightarrow |i-j| \geq e(uv) & (uv \in E,\, u \neq v), \\ i, j \in f(u) \text{ and } i \neq j \Rightarrow |i-j| \geq s(u) & (u \in V). \end{array}$$

The **span** $S(f)$ of a channel assignment f of a constrained weighted graph is the difference between the lowest and the highest colour assigned by f, in other words, $S(f) = \max f(V) - \min f(V)$, where $f(V) = \bigcup_{u \in V} f(u)$. The span $S(G, w)$ of a constrained, weighted graph G and a positive integer vector w indexed by the nodes of G is the minimum span of any channel assignment for (G, w).

We will consider complete graphs with constraints that have a special, nested structure. A constrained graph $G = (V, E, s, e)$ is a **nested clique** with parameters (k, u, a), where $k \geq u \geq a$, if $s(v) \geq k$ for all $v \in V$, and V can be partitioned into two sets Q and R such that $e(vw) \geq a$ if $v, w \in R$, and $e(vw) \geq u$ otherwise.

3 Tile Covers

In this paper, we reduce the channel assignment problem for nested cliques to a tile covering problem. The tiles that may be used for a tile cover are defined in this section. We can think of these tiles as partial assignments, or 'building blocks', from which any possible assignment can be constructed.

We assume that a particular nested clique G with node partition (Q, R) and parameters (k, u, a) is given. We define the set $\mathcal{T}$ of all possible tiles that may be used in a tile cover of G. All tiles are defined as vectors indexed by the nodes of G. For reasons of brevity we will sometimes identify a tile with its support, and thus think of tiles as node sets. It is this representation that allows mention of 'the nodes in tile t'.

In order to facilitate the definition and the proof of Theorem 5.1, we distinguish various categories of tiles. So

$$\mathcal{T} = \mathcal{T}_Q \cup \mathcal{T}_{QR} \cup \mathcal{T}_R \cup \mathcal{T}_{QR}^{big}.$$

The tiles in each category are defined below.

$$\begin{array}{lcl} \mathcal{T}_Q & = & \{\chi^A : A \subseteq Q\}, \\ \mathcal{T}_{QR} & = & \{\chi^A + \chi^B : A \subseteq Q,\, B \subseteq R \text{ where } A \neq \emptyset,\, B \neq \emptyset\} \\ \mathcal{T}_R & = & \{\chi^B : B \subseteq R\}, \\ \mathcal{T}_{QR}^{big} & = & \{\chi^{A \cup B} + \chi^{A_2 \cup B_2} : A_2 \subseteq A \subseteq Q,\, B_2 \subseteq B \subseteq R,\, A_2 \neq \emptyset,\, B_2 \neq \emptyset\} \end{array}$$

The tiles in $\mathcal{T}_{QR}^{big}$ will be called **big** tiles.

A **tiling** is a collection of tiles from $\mathcal{T}$ (multiplicities are allowed). We represent a tiling by a non-negative integer vector $y \in \mathbb{Z}_+^{\mathcal{T}}$, where $y(t)$ represents the number of copies of tile t present in the tiling. A **tile cover** of a weighted nested clique (G, w) is a tiling y such that $\sum_{t \in \mathcal{T}} y(t)t(v) \geq w(v)$ for each node v of G.

With each tile $t \in \mathcal{T}$ we associate a cost $c(t)$. The costs of the tiles in each category are given in Table 3. The cost of each tile t is derived from the minimal span of a channel assignment for (G, t) plus a 'link-up' cost of connecting the assignment to a following tile. This 'link-up' cost is calculated using the assumption that the same assignment will be repeated. For example, let $t = \chi^A$, where $A = \{v_1, \ldots, v_k, v_{k+1}\}$ is a tile of $k+1$ distinct vertices in Q. Then the minimum span of (G, t) is ku, and an assignment of minimum span would be $f(v_i) = iu$ for all i. However, if this assignment is repeated, the next channel that can be assigned will be $(k+2)u$, which is u more than the highest channel in the assignment. Hence the "link-up" cost of this assignment equals u.

In other words, the cost of a tile t is such that for any constant α the minimum span of $(G, \alpha t)$ equals $\alpha c(t)$ minus a small constant, or

$$\frac{S(G, \alpha t)}{\alpha} \to c(t) \text{ as } \alpha \to \infty.$$

It will follow from Theorem 5.1 that our choice of the costs is justified.

Category	Number of nodes in Q	Number of nodes in R	Cost
$\mathcal{T}_Q$	n	0	$\max\{k, nu\}$
$\mathcal{T}_{QR}$	n	m	$\max\{k, nu + ma + u - a\}$
$\mathcal{T}_R$	0	m	$\max\{k, ma\}$
$\mathcal{T}_{QR}^{big}$	n, of which n_2 have weight 2	m, of which m_2 have weight 2	$\max\{k, nu\} + \max\{k, ma\}$ $n_2 u + m_2 a + u - a$

Table 1: Costs of tiles

The **cost** of a tiling y, denoted by $c(y)$, is the sum of the cost of the tiles in the tiling. So $c(y) = \sum_{t \in \mathcal{T}} y(t)c(t)$. The minimal cost of a tile cover of a weighted nested clique (G, w) will be denoted by $\tau(G, w)$.

4 Polyhedral Bounds from Tile Covers

In Section 5 we will prove the following theorem.

Theorem 5.1. *Let G be a nested clique with node partition (Q, R) and parameters (k, u, a). Then for any weight vector w for G,*

$$S(G, w) \geq \tau(G, w) - k.$$

In this section, we will use polyhedral methods to obtain lower bounds for $\tau(G, w)$, and thus for $S(G, w)$.

The problem of finding a minimum cost tile cover of (G, w) can be formulated as an integer program (IP):

$$\begin{array}{ll} \text{Minimize } \sum_{t\in\mathcal{T}} c(t)y(t) & \\ \text{subject to:} & \\ \quad \sum_{t\in\mathcal{T}} t(v)y(t) \geq w(v) & (v \in V) \\ \quad y(t) \geq 0 & (t \in \mathcal{T}) \\ \quad y \text{ integer} & \end{array}$$

We obtain the linear programming (LP) relaxation of this IP by removing the requirement that y must be integral. Any feasible solution to the resulting linear program is called a **fractional** tile cover. The minimum cost of a fractional tile cover gives a lower bound on the minimum cost of a tile cover. The dual of this LP is formulated as follows.

$$\begin{array}{ll} \text{Maximize } \sum_{v\in V} w(v)x(v) & \\ \text{subject to:} & \\ \quad \sum_{v\in V} t(v)x(v) \leq c(t) & (t \in \mathcal{T}) \\ \quad x(v) \geq 0 & (v \in V) \end{array}$$

By linear programming duality, the maximum of the dual is equal to the minimum cost of a fractional tile cover. Thus, any vector that satisfies the inequalities of the dual program gives a lower bound on the cost of a minimum fractional tile cover, and therefore also on the cyclic span of the corresponding complete constrained, weighted graph. The maximum is achieved by one of the vertices of the polytope $TC(G)$ defined as follows:

$$TC(G) = \{x \in \mathbb{Q}_+^V : \sum_{v\in V} t(v)x(v) \leq c(t) \text{ for all } t \in \mathcal{T}\}.$$

A classification of the vertices of this polytope will therefore lead to a comprehensive set of lower bounds that can be obtained from fractional tile covers. For any specific nested clique, such a classification can be obtained by using vertex enumeration software, for example the package `lrs`, developed by David Avis [1]. In general, we can use the dual program to obtain families of vertices, and hence bounds, for certain choices of parameters. To demonstrate the strength of the tile cover approach, in the following theorems we will give such families of bounds for nested cliques with parameters $(k, u, 1)$.

Theorem 4.1 *Let G be a nested clique with node partition (Q, R) and parameters $(k, u, 1)$. Let $w \in \mathbb{Z}_+^V$ be a weight vector for G, and let w_{Qmax} be the maximum weight of any node in Q, and w_{Rmax} maximum weight of any node in R. Then*

$$\tau(G,w) \geq (a-b)w_{Qmax} + b\sum_{v\in Q} w(v) + (c-d)w_{Rmax} + d\sum_{v\in R} w(v),$$

for each 4-tuple (a,b,c,d), where a, b, c and d can take the following values:

a	b	c	d
k	0	0	0
0	0	k	0
$k-(\mu-1)\delta$	δ	δ	0
δ	δ	$k-(\mu-1)\delta$	0
$k-(\mu-1)\delta$	δ	ϵ	ϵ
u	u	1	1
u	u	u	$\frac{k-u}{k-1}$
$2u-1$	ν	1	1

where $\mu = \lfloor \frac{k}{u} \rfloor$, $\delta = (\mu+1)u - k$, $\epsilon = \min\{\frac{\delta}{k-2u+1}, \frac{2u+\mu\delta-\mu}{k+1}\}$, *and* $\nu = u - \max\{\frac{u-1}{\mu}, \frac{\delta-1}{\mu-1}\}$.

Proof. For the proof we consider feasible points in $TC(G)$ that are of the form $a\chi^{\{q\}} + b\chi^{Q-\{q\}} + c\chi^{\{r\}} + d\chi^{R-\{r\}}$, where $q \in Q$ and $r \in R$, and $a \geq b$, $c \geq d$.

For such points, the inequality system that defines $TC(G)$ reduces to the following form:

$$\begin{aligned}
&a + (\mu-1)b \leq k\\
&a + \mu b \leq (\mu+1)u\\
&c + (k-1)d \leq k\\
&a + (n-1)b + c + (m-1)d \leq \max\{k, nu + m + u - 1\} \text{ for all } m,n > 0\\
&2a + (\mu-1)b + 2c + (k-1)d \leq 2k + 2u\\
&2a + \mu b + 2c + (k-1)d \leq k + (\mu+3)u\\
&a,b,c,d \geq 0
\end{aligned}$$

From the second and the third inequality, derived from a tile from $\mathcal{T}_Q$ of size $\mu+1$, and a tile from $\mathcal{T}_R$ of size k, respectively, we see that $b \leq u$ and $d \leq 1$. Since the cost of a tile never increases by more than u if a node from Q is added, and by 1 if a node from R is added, the inequalities that correspond to tiles with more than $\mu+1$ nodes in Q or more than k nodes in R are implied by the inequalities above.

It is straightforward to check that all values for (a,b,c,d) given in the theorem give feasible points of this system. Note that each element satisfies at least four of the inequalities with equality. So for each vector $x = a\chi^{\{q\}} + b\chi^{Q-\{q\}} + c\chi^{\{r\}} + d\chi^{R-\{r\}}$ with $(a,b,c,d) \in \xi$ and q and r any nodes in Q and R, respectively, it holds that

$x \in TC(G)$, and thus $\tau(G,w) \geq \sum_{v\in V} w(v)x(v)$. Since $a \geq b$ and $c \geq d$, for each such vector x, $\sum_{v\in V} w(v)x(v)$ is maximized when we choose q and r to be the nodes of maximum weight in Q and R, respectively. With this choice of q and r, $\sum_{v\in V} w(v)x(v) = (a-b)w_{Qmax} + b\sum_{v\in Q} w(v) + (c-d)w_{Rmax} + d\sum_{v\in R} w(v)$, so the bound follows. □

The bound derived from $(k,0,0,0)$ and $(0,0,k,0)$ is the well known bound

$$S(G,w) \geq kw_{max} - k.$$

The bound from $(u,u,1,1)$ was derived in [4] using ad hoc methods. In the same paper, a bound $S(G,w) \geq (k-\mu\delta)w_{max} + \delta\sum_{v\in C} w(v) - \kappa$ (κ is a small constant) was given for cliques with co-site constraint k and uniform edge constraint u. The points $(k-(\mu-1)\delta, \delta, \delta, 0)$ and $(\delta, \delta, k-(\mu-1)\delta, 0)$ also give this bound, because $Q \cup \{v_{Rmax}\}$ forms a clique with edge constraint at least u:

$$S(G,w) \geq (k-\mu\delta)w_{Qmax} + \delta(\sum_{v\in Q} w(v) + w_{Rmax}) - k.$$

The bound derived from $(k-(\mu-1)\delta, \delta, \epsilon, \epsilon)$ can also be seen as generalizations and extensions of the mentioned clique bound, here we have

$$S(G,w) \geq (k-\mu\delta)w_{Qmax} + \delta\sum_{v\in Q} w(v) + \epsilon\sum_{v\in R} w(v) - k.$$

From $(u,u,u,\frac{k-u}{k-1})$ we obtain the bound

$$S(G,w) \geq u(\sum_{v\in Q} w(v) + w_{R\max}) + \frac{k-u}{k-1}\sum_{v\in R, v\neq v_{Rmax}} w(v) - k.$$

This bound strengthens the bound $u\sum_{v\in C} w(v) - u$, which holds for any clique C with edge constraint u.

In [13] a bound of $(2u-1)w_{Qmax} + \sum_{v\in R} w(v) - \kappa$ is given for nested cliques with the special property that $|Q| = 1$. Using our methods, we see that this bound extends to nested cliques where Q contains more than one node. Using the point $(2u-1, \nu, 1, 1)$ we obtain

$$S(G,w) \geq (2u-1)w_{Qmax} + \nu\sum_{v\in Q, v\neq v_{Qmax}} w(v) + \sum_{v\in R} w(v) - k.$$

In all these results, we have used the general rule that $S(G,w) \geq \tau(G,w) - k$. We will see in the next section that the additive term k comes from removing 'patches' from the tiling. However, in most cases the extra term k will be too pessimistic. In principle, it is possible to find a more precise additive term by taking the patches into account while formulating the LP. We felt that this would make the analysis unduly complicated, especially since we were mainly interested in showing a method by which lower bounds can be derived. However, the reader should keep this in mind as an explanation as to why our bounds may differ slightly from the older bounds.

The following theorem gives two new bounds for another variation of the parameters (k, u, a).

Theorem 4.2 *Let G be a nested clique with node partition (Q, R) and parameters (k, u, a), where $\lfloor \frac{k}{u} \rfloor = 1$ and $\lfloor \frac{k}{a} \rfloor = 2$. Let $w \in \mathbb{Z}_+^V$ be a weight vector for G, and let w_{Qmax} be the maximum weight of any node in Q, and w_{Rmax} maximum weight of any node in R. Then*

$$S(G, w)) \geq (3u - k + 2\delta) \sum_{v \in Q} w(v) + (k - 2\delta) w_{Rmax} + \delta \sum_{v \in R} w(v) - k,$$

where $\delta = 3a - k$, and

$$S(G, w) \geq u(\sum_{v \in V} w(v) + w_{Rmax}) + \frac{3a - u}{2} \sum_{v \in R, v \neq v_{Rmax}} w(v) - k.$$

Proof. For the proof we again consider feasible points in $TC(G)$ that are of the form $a\chi^{\{q\}} + b\chi^{Q-\{q\}} + c\chi^{\{r\}} + d\chi^{R-\{r\}}$, where $q \in Q$ and $r \in R$, and $a \geq b$, $c \geq d$.

For such points, and for parameters as mentioned in the theorem, the inequality system that defines $TC(G)$ reduces to the following form:

$a \leq k$
$a + b \leq 2u$
$c + d \leq k$
$c + 2d \leq 3a$
$a + c \leq 2u$ for all $m, n > 0$ $a, b, c, d \geq 0$

It is straightforward to check that $(u, u, u, \frac{3a-u}{2})$ and $(3u-k+\delta, 3u-k+\delta, k-\delta, \delta)$ are feasible points of this system. When we choose the vectors of this form so that the maximum coefficients correspond to the nodes of maximum weight, the bounds follow. □

The preceeding theorems show how new lower bounds can be generated for any particular choice of parameters. In practice, it will often be useful to apply the tile cover method directly to the exact parameters of the particular network. This approach is demonstrated in the following example. The example is taken from [15], where it was used to demonstrate a lower bound derived from network flows. We will see that our tile cover approach gives a significant improvement.

Example 4.1 *Consider the cellular network layout as shown Figure 4.1. The circled numbers in each cell represent the label of the cell; the node associated with the cell with label i is called v_i. The larger number in each cell gives the demand in the cell, i.e. the weight of the associated node. The particular hexagonal cell layout of this example has been frequently used as a benchmark for algorithms and lower bounds for the channel assignment problem (see for example [4],[11],[16],[10],[5]).*

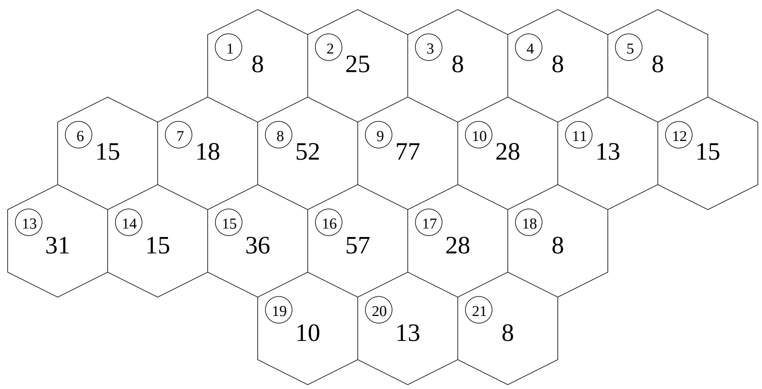

Figure 1: The layout of the example.

The constraints are described in terms of the distance d_{ij} between the centers of cells v_i and v_j where the unit is the distance between the centers of adjacent cells.

$$c_{ij} = \begin{cases} 0 & \text{if } d_{ij} > 3, \\ 1 & \text{if } \sqrt{3} < d_{ij} \leq 3, \\ 2 & \text{if } 0 < d_{ij} \leq \sqrt{3}, \\ 5 & \text{if } i = j \end{cases}$$

This layout contains nested cliques of size 8, with 2 nodes in Q and 6 nodes in R, and nested cliques of size 7, with one node in Q and 6 *nodes in R. The nested cliques have parameters* $(5, 2, 1)$.

For a nested clique with bipartition (Q, R) where $|Q| = 2$ and $|R| = 6$, we derived a set of lower bounds using the software `lrs`. *We looked for points of the form $(x_1, x_2, y_1, y_2, y_3, y_4, y_6, y_6)$, where x_1 and x_2 correspond to nodes of Q and $x_1 \geq x_2$, and $y_1, \ldots, y_6$ correspond to the nodes of R, and $y_1 \geq y_2 \geq \ldots \geq y_6$. The inequality system that defines $TC(G)$ reduces to the following:*

$x_1 + x_2 \leq 5$
$x_1 + x_2 + y_1 \leq 6$
$y_1 + y_2 + y_3 + y_4 + y_5 \leq 5$
$x_1 + y_1 + y_2 \leq 5$
$2x_1 + x_2 + 2y_1 + y_2 + y_3 + y_4 + y_5 \leq 14$
$x_1 \geq x_2,\ y_1 \geq y_2 \geq \ldots, \geq y_6$
$x_1, x_2, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$

Given this system, `lrs` *returned a set of vertices, 14 of which could be used to generate lower bounds (the other vertices could be obtained from those 14 by dropping some coordinates to zero).*

We applied these bounds to the nested clique formed by the cells as indicated in Figure 1. Here $Q = \{v_9, v_{16}\}$*, and* $R = \{v_2, v_8, v_{10}, v_{15}, v_{17}, v_{20}\}$*. To obtain best possible results, the nodes of larger weight in* Q *and* R *where matched with larger coordinates* x_i *or* y_i*, respectively. The best result was obtained by the point* $(3, 2, 1, 1, 1, 1, 1, 1)$*. The corresponding lower bound is*

$$\begin{aligned} S(G, w) &\geq 3w(v_9) + 2w(v_{16}) + \sum_{v \in R} w(v) - 5 \\ &= 3 \cdot 77 + 2 \cdot 57 + (52 + 36 + 28 + 28 + 25 + 13) - 5 \\ &= 522. \end{aligned}$$

This improves by 13% the lower bound of 460 obtained in [15].

5 From Channel Assignments to Tile Covers

In this section we give the proof of the theorem:

Theorem 5.1 *Let* G *be a nested clique with node partition* (Q, R) *and parameters* (k, u, a)*. Then for any weight vector* w *for* G*,*

$$S(G, w) \geq \tau(G, w) - k.$$

This theorem will follow as a corollary from a more technical lemma. The lemma reduces any channel assignment to a tiling that uses tiles from $\mathcal{T}$, but also at most one extra tile, added to take care of the highest channels assigned, for which there is no 'link-up' cost. These extra tiles will be called patches, and are defined as follows.

Given is a nested clique G with node bipartition (Q, R) and constraints $(k, u, 1)$. The patch set $\mathcal{P}$ is defined as follows.

$$\mathcal{P} = \mathcal{P}_Q \cup \mathcal{P}_{QR} \cup \mathcal{P}_R \cup \mathcal{P}_{QR}^{big}.$$

The patches in each category are defined below.

$$
\begin{array}{lcl}
\mathcal{P}_Q & = & \{\chi^A : A \subseteq Q \text{ and } 1 \le |A| \le \lfloor \frac{k}{u} \rfloor + 1\} \\
\mathcal{P}_R & = & \{\chi^B : B \subseteq R,\, 1 \le |B| \le k\}, \\
\mathcal{P}_{QR} & = & \{\chi^A + \chi^B : A \subseteq Q,\, B \subseteq R,\, 1 \le |A| \le \lfloor \frac{k}{u} \rfloor + 1,\, 1 \le |B| \le k\}. \\
\mathcal{P}^{big}_{QR} & = & \{\chi^{A \cup B} + \chi^{A_2 \cup B_2} : A_2 \subseteq A \subseteq Q,\, B_2 \subseteq B \subseteq R,\, A_2 \ne \emptyset,\, B_2 \ne \emptyset\}
\end{array}
$$

The costs of the patches in each category are given in Table 5. It is straightforward to check that, in fact, the cost of each patch p corresponds to the minimal cost of a channel assignment for (G, p).

Category	Number of nodes in Q	Number of nodes in R	Cost
$\mathcal{P}_Q$	n	0	$(n-1)u$
$\mathcal{P}_R$	0	m	$(m-1)a$
$\mathcal{P}_{QR}$	n	m	$nu + (m-1)a$
$\mathcal{P}^{big}_{QR}$	n, of which n_2 have weight 2	m, of which m_2 have weight 2	$(n+n_2)u + (m_2-1)a+$ $\max\{k, ma\}$

Table 2: Costs of patches

When we reduce a channel assignment to a tiling, the patches in $\mathcal{P}_R$ will only be used when the first channel is assigned to a node in R, and the patches in $\mathcal{P}_Q$ and $\mathcal{P}^{big}_{QR}$ will only be used if the first channel is assigned in Q. Before we prove and state the general lemma, we first prove a lemma that reduces channel assignments to tilings for cliques with only one edge constraint. This can be seen as a lemma that applies to nested cliques with node bipartition (Q, R), and the special property that $Q = \emptyset$ or $R = \emptyset$ (in the latter case, we take u to be equal to a).

For the rest of this section we will adopt the following terminology. If a channel assignment f for a constrained graph G with node set V is given, and if $f(V) = \{c_0, c_1, \ldots, c_f\}$, with $c_0 \le c_1 \le \ldots \le c_f$, then we say that a tiling y for G **covers** channels c_i to c_j (where $j \ge i$), if this tiling covers as much weight as the assignment restricted to the channels between c_i and c_j. More precisely, y covers channels $\{c_i, \ldots, c_j\}$ if for each node $v \in V$, $\sum_{t \in \mathcal{T}} y(t)t(v) \ge |f(v) \cap \{c_i, \ldots, c_j\}|$.

Also, when y is a tiling and t is a patch or tile, we use $y + \{t\}$ to mean the tiling where one more copy of t is added, so, strictly speaking, the tiling $y + \chi^{\{t\}}$.

We start by stating a lemma that proves that any channel assignment can be reduced to a tile cover for the cliques where there is only one edge constraint, and a co-site constraint. This lemma was proved in [8], but we restate the proof, both because we need a slightly extended version of the lemma in our later proof, and because the proof of this lemma gives the flavour of the later proof.

Lemma 5.2 *[8] Let G be a clique with co-site constraint k and edge constraint u. Let Q be the node set of G, and let the tile set $\mathcal{T}_Q$ and patch set $\mathcal{P}_Q$ be as defined above. Then for any channel assignment of (G, w) of span s there exists a tile cover $y \in \mathbb{Z}^{\mathcal{T}_Q \cup \mathcal{P}_Q}$, which contains exactly one patch, p, of (G, w) with cost at most s. Moreover, the support of p consists of the nodes that receive the last $|V(p)|$ channels of the assignment.*

Proof. Let f be a channel assignment of (G, w) of span s, using channels $\{c_0, c_1, \ldots, c_f\}$, where $c_0 < c_1 < \ldots < c_f$. Let $\mu = \lfloor \frac{k}{u} \rfloor$. We will construct the required tile cover tile by tile.

For all j, $0 \leq j \leq f$, let the partial tile cover y_j denote a collection of tiles (no patches) such that $c(y_j) \leq c_j - c_0$ and y_j covers channels $\{c_0, \ldots, c_{j-1}\}$. We start the construction of the tile cover with the empty tile collection $y_0 = 0$, so $c(y_0) = 0 = c_0 - c_0$. Next, supposing that we already have a partial tile cover y_j, we proceed to construct a new family $y_{j'}$ for some higher value $j' > j$.

(i) If any node of G receives a channel $c_{j'}$ such that $c_j + k \leq c_{j'} < c_j + (\mu + 1)u$, then since G is a clique and because channels on neighbouring nodes have to differ by at least u, no node in G has a channel in the interval $[c_j + \mu u, c_j + k)$. Let A be the set that contains all nodes of G with a channel in the range $[c_j, c_j + \mu u)$, and let the tile $t = \chi^A$; t covers all channels between c_j and $c_{j'-1}$. Since A can contain at most μ nodes, t has cost k. Let $y_{j'} = y_j + \{t\}$, then $c(y_{j'}) = c(y_j) + c(t) \leq c_j - c_0 + k \leq c_{j'} - c_0$.

(ii) If (i) fails and $c_f \geq c_j + (\mu + 1)u$, then no node can have *two* channels from the range $[c_j, \ldots, c_j + (\mu + 1)u)$, because of the requirement that channels on the same node have to differ by at least k. Let $c_{j'}$ be the least channel such that $c_{j'} \geq c_j + (\mu + 1)u$. Let A be the set that contains all nodes of G with a channel in the range $[c_j, c_j + (\mu + 1)u)$, and let tile $t = \chi^A$; t covers all channels between c_j and $c_{j'-1}$. Since A can contain at most $\mu + 1$ nodes, t has cost at most $(\mu + 1)u$. Let $y_{j'} = y_j + \{t\}$, then $c(y_{j'}) = c(y_j) + c(t) \leq c_j - c_0 + (\mu + 1)u \leq c_{j'} - c_0$.

(iii) If (i) fails and $c_f < c_j + (\mu + 1)u$, then we conclude the construction by adding one patch, $p = \chi^A$, where A is the set which consists of the nodes of G with a channel in the range $[c_j, c_f]$. Since $c_f < c_j + (\mu + 1)u$, $|A| \leq \mu + 1$, and because (i) fails, no node can have two channels from $\{c_j, \ldots, c_f\}$. Take $y = y_j + \{p\}$. Now $c_f \geq c_j + (|A| - 1)u$, and thus $c(y) = c(y_j) + c(p) = c_j - c_0 + (|A| - 1)u \leq c_f - c_0 = s$. Therefore, y is the desired tile cover. □

In fact, it is possible to strengthen the lemma and show that for a clique with uniform co-site constraint k and edge constraint u, for any tile cover of cost s, there exists a channel assignment that covers the same weights of span at most s. By using tile covers of this kind, bounds can be obtained from a polyhedral analysis in the same way as was done in the previous sections. For a complete discussion of this case, see [7], and for a synopsis see [8].

We are now ready to state and prove the technical lemma from which Theorem 5.1 will follow. The proof of this lemma is based on a straightforward induction, but unfortunately becomes rather long because of a number of cases that depend on how the patches of inductively obtained tilings must be combined.

The induction is on the number of times the channel assignment "crosses over" from Q to R or vice versa. The base case can be directly derived from Lemma 5.2. For the induction step, three different tilings are obtained. By invocation of Lemma 5.2, tilings are obtained for the first parts of the channel assignment up to the first crossover and between the first and second crossover, respectively. Then induction is used to obtain a tiling of the channel assignment after the second crossover. These three tilings are then combined to obtain one new tiling which satisfies the induction hypothesis. The difficulties arise mainly from the fact that three different patches must be combined. Because of the different types of patches, there are a number of cases that must be considered.

Lemma 5.3 *Let G be a nested clique with node partition (Q, R) and constraints (k, u, a), and let $\mathcal{T}$ and $\mathcal{P}$ be the tile and patch set for G. Let f be a channel assignment for G, where $f(V) = \{c_0, c_1, \ldots, c_f\}$, $c_0 < c_1 < \ldots, c_f$. Then there exists a tile cover $y \in \mathbb{Z}_+^{\mathcal{T}\cup\mathcal{P}}$ of (G, w) which contains at most one patch p, covers all channels $\{c_0, \ldots, c_f\}$, and has cost at most $c_f - c_0$.*

Furthermore, if c_0 is assigned to a node in Q then $p \notin \mathcal{P}_R$, and if c_0 is assigned to a node in R then $p \notin \mathcal{P}_Q \cup \mathcal{P}_{QR}^{big}$.

Proof. Let G be a nested clique as defined in the statement of the lemma. We will prove the lemma by induction on the number of *crossovers* of the channel assignment. A crossover is a pair of channels (c_i, c_{i+1}) where the nodes that receive channels c_i and c_{i+1} are in different part of the bipartition (Q, R).

If f is a channel assignment for G with no crossovers, then the lemma follows directly from Lemma 5.2. Let f be a channel assignment with one crossover, and $f(V) = \{c_0, c_1, \ldots, c_f\}$, where $c_0 < c_1 < \ldots < c_f$.

When c_0 is in Q, let c_ℓ be the first colour in R greater than c_0. By lemma 5.2, we can cover the channels $\{c_0, \cdots, c_{\ell-1}\}$ with a tiling y_Q containing one patch $p_Q \in \mathcal{P}_Q$ and with cost at most $c_{\ell-1} - c_0$, and the channels $\{c_\ell, \cdots, c_f\}$ with a tiling y_R containing one patch P_R and with cost not more than $c_f - c_\ell$. Combining the two patches into one, we form a new patch $p' = p_Q + p_R \in \mathcal{P}_{QR}$ with cost $nu + (m-1)a$, where n is the number of nodes in p_q and m is the number of nodes in p_R, so $c(p') = c(p_Q) + c(p_R) + u$. Our final tiling is $y = y_Q + y_R - \{p_Q\} - \{p_R\} + \{p'\}$ with cost

$$\begin{aligned} c(y) &= c(y_Q) + c(y_R) - c(p_Q) - c(p_R) + c(p') \\ &\leq (c_{\ell-1} - c_0) + (c_f - c_\ell) + u \\ &\leq c_{\ell-1} - c_0 + c_f - (c_{\ell-1} + u) + u \\ &= c_f - c_0. \end{aligned}$$

When c_0 is in R, the proof is analogous.

For the induction step, let f be a channel assignment with g crossovers, where $g \geq 2$, and assume that the lemma holds for any channel assignment with less than g crossovers. Let $f(V) = \{c_0, c_1, \ldots, c_f\}$, where $c_0 < c_1 < \ldots < c_f$.

CASE 1: Channel c_0 is assigned to a node in Q. Let c_ℓ be the first channel assigned to a node in R, and c_j the first channel greater than c_ℓ assigned to a node in Q. So $(c_{\ell-1}, c_\ell)$ and (c_{j-1}, c_j) are the first two crossovers of f. Note that $c_\ell \geq c_{\ell-1} + u$ and $c_j \geq c_{j-1} + u$.

By lemma 5.2, we can find a tiling y_Q (with one patch, $p_Q \in \mathcal{P}_Q$) which covers channels $\{c_0, \cdots, c_{\ell-1}\}$ in Q and has cost at most $c_{\ell-1} - c_0$, and a tiling y_R (with one patch, $p_R \in \mathcal{P}_R$) which covers channels $\{c_\ell, \ldots c_{j-1}\}$ and has cost at most $c_{j-1} - c_\ell$.

Let $|V(p_Q)| = n, |V(p_R)| = m$. Then the cost of p_Q is $(n-1)u$ and the cost of p_R is $(m-1)a$. By Lemma 5.2, $V(p_Q)$ consists of the nodes that receive channels $\{c_{\ell-n}, \ldots, c_{\ell-1}\}$. Note that, since all channels $\{c_{\ell-n}, \ldots, c_{\ell-1}\}$ are assigned to nodes in Q, $c_{\ell-1} \geq c_{\ell-n} + (n-1)u$. Therefore, $y_Q - \{p_Q\}$ covers channels $\{c_0, \ldots, c_{\ell-n-1}$ and has cost at most $c_{\ell-n} - c_0$.

Case 1A. Tiling y_R contains no tiles (only the patch p_R).

In this case, y_R only contains the patch p_R, which covers channels c_ℓ to $c_{\ell+m-1}$. Therefore, $m = j - \ell$, and p_R covers all channels from c_ℓ to c_{j-1}.

If $c_j \geq c_{\ell-n} + k$, then we form a tile $t' = p_Q + p_R \in \mathcal{T}_{QR}$. Since the channels from $c_{\ell-n}$ to c_{j-1} cover n nodes in Q and m nodes in R, and contain at least two crossovers, we have that $c_{j-1} \geq c_{\ell-n} + (n-1)u + (m-1)a + 2u$, and thus $c_j - c_{\ell-n} \geq \max\{nu + ma + u - a, k\} = c(t')$.

The assignment restricted to channels $\{c_j, \ldots, c_f\}$ has $g-2$ crossovers, so by induction, there exist a tiling y_{end} which covers channels $\{c_j, \ldots c_f\}$ and has cost at most $c_f - c_j$. Also, since the first channel c_j was assigned to a node of Q, the patch of y_{end} is not from $\mathcal{P}_R$. Let $y = y_Q - \{p_Q\} + \{t'\} + y_{end}$. Then y covers all channels, and

$$
\begin{aligned}
c(y) &= c(y_Q - \{p_Q\}) + c(t') + c(y_{end}) \\
&\leq (c_{\ell-n} - c_0) + (c_j - c_{\ell-n}) + (c_f - c_0) \\
&= c_f - c_0.
\end{aligned}
$$

Suppose then that $c_j < c_{\ell-n} + k$. If any channel c_i in the range $[c_{\ell-n} + k, c_{\ell-n} + k + u)$ has been assigned to a node in Q, then let w be the node to which this channel is assigned (since the range has length u, there can be at most one such channel and node). Let A be the set containing all nodes that receive channels $\{c_{\ell-n}, \dots, c_{i-1}\}$. Note that no two channels from $\{c_{\ell-n}, \dots, c_{i-1}\}$ can be assigned to the same node, because the co-site constraint on any node is at least k, and $c_{i-1} \leq c_i - u < c_{\ell-n+k}$.

Let $n_1 = |A \cap Q|$ and $m_1 = |A \cap R|$. Then, since the channels $\{c_{\ell-n}, \cdots, c_{i-1}\}$ must cover n_1 nodes from Q and m_1 nodes from R and contain at least two crossovers, $c_i \geq c_{\ell-n} + n_1 u + (m_1 - 1)a + u$. Since $c_i \geq c_{\ell-n} + k$ by the definition of c_i, we can now state that $c_i \geq c_{\ell-n} + \max\{k, n_1 u + m_1 a + u - a\}$. We make a new tile $t' = \chi^A \in \mathcal{T}_{QR}$, of cost $c(t') = \max\{k, n_1 u + m_1 a + u - a\} \leq c_i - c_{\ell-n}$.

By induction, there exists a tiling y_{end} that covers channels $\{c_i, \dots, c_f\}$ and has cost at most $c_f - c_i$. Let $y = y_Q - \{p_Q\} + \{t'\} + y_{end}$. Tiling y covers all channels, and has cost

$$
\begin{aligned}
c(y) &= c(y_Q - \{p_Q\}) + c(t') + c(y_{end}) \\
&\leq (c_{\ell-n} - c_0) + (c_i - c_{\ell-n}) + (c_f - c_i) \\
&= c_f - c_0
\end{aligned}
$$

If no channel from the range $[c_{\ell-n} + k, c_{\ell-n} + k + u)$ is assigned to a node in Q, then let c_i be the first channel greater than or equal to $c_{\ell-n} + k + u$. If there is no such channel, so if $c_f < c_{\ell-n} + k$, then let $c_i = c_f$. Let w be the node that channel c_i is assigned to. By induction, there exists a tiling y_{end} of cost at most $c_f - c_i$ that covers channels $\{c_i, \dots, c_f\}$. Let p be the patch of y_{end}.

Let A be the set of nodes that receive channels $\{c_{\ell-n}, \dots, c_{i-1}\}$, and let $n_1 = |Q \cap A|$, and $m_1 = |R \cap A|$. Since the set $\{c_{\ell-n}, \dots, c_i\}$ contains at least two crossovers, we have that $c_i \geq c_{\ell-n} + n_1 u + m_1 a + u - a$.

If $w \in Q$, then make a tile $t' = \chi^A \in \mathcal{T}_{QR}$, which will have cost $\max\{k, n_1 u + m_1 a + u - a\} \leq c_i - c_{\ell-n}$. Let $y = y_Q - \{p_Q\} + \{t'\} + y_{end}$. Then tiling y covers all channels, has a patch of the right type (since the assignment restricted to channels $\{c_i, \dots, c_f\}$ also has its first channel assigned to a node in Q, so p is of the right type), and has cost

$$
\begin{aligned}
c(y) &= c(y_Q - \{p_Q\}) + c(t') + c(y_{end}) \\
&\leq (c_{\ell-n} - c_0) + (c_i - c_{\ell-n}) + (c_f - c_i) \\
&= c_f - c_0.
\end{aligned}
$$

If $w \in R$, then it may be that p is in $\mathcal{P}_R$, which is not allowed. If this happens, extend patch p by adding a node $v \in A_Q$. So $p' = p + \chi^{\{v\}} \in \mathcal{P}_{QR}$, and has cost $c(p') = c(p) + u$. Make a tile $t' = \chi^A - \chi^{\{v\}}$. If $n_1 > 1$ then $t' \in \mathcal{T}_{QR}$ and $c(t') = \max\{k, (n_1 - 1)u + m_1 a + u - a\}$, and if $n_1 = 1$ then $t \in \mathcal{T}_R$ and $c(t') = \max\{k, (m_1 - 1)a\}$.

Now $c_i \geq c_{\ell-n} + n_1 u + m_1 a + u - a$ and $c_i \geq c_{\ell-n} + k + u$. So $c(t') \leq c_i - c_{\ell-n} - u$. Let $y = y_Q - \{p_Q\} + y_{end} - \{p\} + \{p'\} + \{t'\}$. The tiling y covers all channels, has a patch of the right type, and has cost

$$
\begin{aligned}
c(y) &= c(y_Q - \{p_Q\}) + c(y_{end}) - c(p) + c(p') + c(t') \\
&\leq (c_{\ell-n} - c_0) + (c_f - c_i) + u + (c_i - c_{\ell-n} - u) \\
&\leq (c_{\ell-n} - c_0) + (c_f - c_i) + (c_i - c_{\ell-n}) \\
&= c_f - c_0.
\end{aligned}
$$

Case 1B. y_R contains tiles.

By Lemma 5.2, patch p_R covers channels $\{c_{j-m}, \ldots, c_{j-1}\}$, and these channels are all assigned to nodes in R, so $j - m \geq \ell$. Since $(c_{\ell-1}, c_\ell)$ is a crossover, the assignment of channels $\{c_{j-m}, \ldots, c_f\}$ has $g - 1$ crossovers, so by induction there exists a tiling y_{end} that covers channels $\{c_{j-m}, \ldots, c_f\}$.

Let p be the patch of y_{end}, and let A_Q be the set of nodes from Q in the support of p, and let A_R be the set of nodes from R in the support of p. Let $|A_Q| = n_1$ and $|A_R| = m_1$. Since $p \in \mathcal{P}_R \cup \mathcal{P}_{QR}$, $m_1 \geq 1$. Let t be a tile from y_R, so $t \in \mathcal{T}_R$. Let $V_R = V(t)$, the support of t, and $V_Q = V(p_Q)$, the support of p_Q. We will assume without loss of generality that $|V_R| \geq \lfloor \frac{k}{a} \rfloor$.

In the following table, we show how to combine p_Q, p and t into a new tile t' and a new patch p'.

Case	Condition	Tile t'	Patch p'
(1.1)	$p \in \mathcal{P}_{QR}$ and $n_1 u + m_1 + u - 1 \geq k$	p	p_Q
(1.2)	$p \in \mathcal{P}_{QR}$ and $n_1 u + m_1 + u - 1 < k$		
(1.2.1)	(1.2) and $A_Q \cap V_Q = \emptyset$	t	$p + p_Q$
(1.2.2)	(1.2) and $A_Q \cap V_Q \neq \emptyset$		
(1.2.2.1)	(1.2.2) and $A_R \cap V_R = \emptyset$	$p + t$	p_Q
(1.2.2.2)	(1.2.2) and $A_R \cap V_R \neq \emptyset$		
(1.2.2.2.1)	and $\lvert V_Q \cap A_Q \rvert \leq \lfloor \frac{k}{u} \rfloor$	there is no t'	$p_Q + t + p$
(1.2.2.2.2)	and $\lvert V_Q \cap A_Q \rvert > \lfloor \frac{k}{u} \rfloor$	$p_Q + \chi^{A_Q - V_Q}$	$\chi^{A_Q \cap V_Q} + \chi^{A_R}$
(3.1)	$p \in \mathcal{P}_R$ and $m_1 \geq k$	p	p_Q
(3.2)	$p \in \mathcal{P}_R$ and $m_1 < k$	t	$p_Q + p$

Case	Cost $c(t')$	$t' \in$	Cost $c(p')$	$p' \in$
(1.1)	$n_1 u + m_1 a + u - a$	$\mathcal{T}_{PQ}$	$c(p_Q)$	$\mathcal{P}_Q$
(1.2.1)	$c(t)$	$\mathcal{T}_R$	$(n + n_1)u + (m_1 - 1)a$	$\mathcal{P}_{QR}$
(1.2.2.1)	$n_1 u + m_1 a + \lvert V_R \rvert a + u - a$	$\mathcal{T}_{QR}$	$c(p_Q)$	$\mathcal{P}_Q$
(1.2.2.2.1)	–	–	$(n + n_1)u + k + (m_1 - 1)a$	$\mathcal{P}_{QR}^{big}$
(1.2.2.2.2)	$(n + \lvert A_Q - V_Q \rvert)u$	$\mathcal{T}_Q$	$\lvert A_Q \cap V_Q \rvert u + (m_1 - 1)a$	$\mathcal{P}_{QR}$
(3.1)	$n_1 u + m_1 a + u - a$	$\mathcal{T}_R$	$c(p_Q)$	$\mathcal{P}_Q$
(3.2)	$c(t)$	$\mathcal{T}_R$	$(n + n_1)u + (m_1 - 1)a$	$\mathcal{P}_{QR}$

Table 3: Combining patches

In cases (1.2.1), (1.2.2.1) and (3.2), we form the new tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} - \{t\} + y_{end} - \{p\} + \{t'\} + \{p'\}.$$

It can be verified in the table that in all these cases, $c(t') + c(p') - c(p_Q) - c(p) - c(t) \leq u$, so

$$c(y) \quad = \quad c(y_Q) - c(p_Q) + c(y_R) - c(p_R) - c(t) + c(y_{end}) - c(p) + c(t') + c(p')$$

$$
\begin{aligned}
&\leq (c_{\ell-1} - c_0) + (c_{j-1} - c_\ell) - (m-1)a + (c_f - c_{j-m}) + u \\
&\leq (c_{\ell-1} - c_0) + (c_{j-m} - c_\ell) + (c_f - c_{j-m}) + (c_\ell - c_{\ell-1}) \\
&= c_f - c_0.
\end{aligned}
$$

In case (1.2.2.2.1), we take the tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} - \{t\} + y_{end} - \{p\} + \{p'\},$$

with cost

$$
\begin{aligned}
c(y) &= c(y_Q) - c(p_Q) + c(y_R) - c(p_R) - c(t) + c(y_{end}) - c(p) + c(p') \\
&\leq (c_{\ell-1} - c_0) - (n-1)u + (c_{j-1} - c_\ell) - (m-1)a - k \\
&\quad + (c_f - c_{j-m}) - (n_1 u + (m_1 - 1)a) + ((n + n_1)u + k + m_1 a - a) \\
&\leq (c_{\ell-1} - c_0) + (c_{j-m} - c_\ell) + (c_f - c_{j-m}) + u \\
&\leq (c_{\ell-1} - c_0) + (c_{j-m} - c_\ell) + (c_f - c_{j-m}) + (c_\ell - c_{\ell-1}) \\
&= c_f - c_0.
\end{aligned}
$$

In cases (1.1), (3.1) and (1.2.2.2.2), t is not used, so we have the tiling

$$y = y_Q - \{p_Q\} + y_R - \{p_R\} + y_{end} - \{p\} + \{t'\} + \{p'\}.$$

It can be verified using the table that in all these cases, $c(t') + c(p') - c(p) - c(p_Q) \leq u$, so

$$
\begin{aligned}
c(y) &= c(y_Q) - c(p_Q) + c(y_R) - c(p_R) + c(y_{end}) - c(p) + c(t') + c(p') \\
&\leq (c_{\ell-1} - c_0) + (c_{j-1} - c_\ell) - (m-1)a + (c_f - c_{j-m}) + u \\
&\leq (c_{\ell-1} - c_0) + (c_{j-m} - c_\ell) + (c_f - c_{j-m}) + (c_\ell - c_{\ell-1}) \\
&= c_f - c_0.
\end{aligned}
$$

In each case, y covers all channels, the patch of y is of the right type, and $c(y) \leq c_f - c_0$, as required.

CASE 2: Channel c_0 is assigned to a node in R.

Let c_ℓ be the first channel assigned to a node in Q, and c_j the first channel greater than c_ℓ assigned to a node in R. So $(c_{\ell-1}, c_\ell)$ and (c_{j-1}, c_j) are the first two crossovers of f. Note that $c_\ell \geq c_{\ell-1} + u$ and $c_j \geq c_{j-1} + u$.

By lemma 5.2, we can find a tiling y_R (with one patch, $p_R \in \mathcal{P}_R$) which covers channels $\{c_0, \cdots, c_{\ell-1}\}$ in R and has cost at most $c_{\ell-1} - c_0$, and a tiling y_Q (with one patch, $p_Q \in \mathcal{P}_Q$) which covers channels $\{c_\ell, \ldots c_{j-1}\}$ and has cost at most $c_{j-1} - c_\ell$.

Let $|V(p_Q)| = n$, $|V(p_R)| = m$. The cost of p_Q is $(n-1)u$ and the cost of p_R is $(m-1)a$. By Lemma 5.2, $V(p_R)$ consists of the nodes that receive channels $\{c_{\ell-m}, \dots, c_{\ell-1}\}$. Also, $c_{\ell-1} \geq c_{\ell-m} + (m-1)a$. Thus $y_R - \{p_R\}$ covers channels $\{c_0, \dots, c_{\ell-m-1}\}$ and has cost at most $c_{\ell-m} - c_0$.

Case 2A. Tiling y_Q contains no tiles (only the patch p_Q).

Since this case is very similar to Case 1A, we only sketch the proof.

If $c_j \geq c_{\ell-m} + k$, then we form a tile $t' = p_Q + p_R \in \mathcal{T}_{QR}$. We have that $c_j \geq c_{\ell-m} + (n-1)u + (m-1)a + 2u$, and thus $c_{\ell-m} - c_j \leq c(t')$. By induction, there exists a tiling y_{end} which covers channels $\{c_j, \dots c_f\}$ and has cost at most $c_f - c_j$, and the patch of y_{end} is not from $\mathcal{P}_Q$ or $\mathcal{P}_{QR}^{big}$. Let $y = y_R - \{p_R\} + \{t'\} + y_{end}$. Then y covers all channels, and

$$c(y) = c(y_R - \{p_R\}) + c(t') + c(y_{end}) \leq c_f - c_0.$$

Suppose then that $c_j < c_{\ell-m} + k$. If any channel in the range $[c_{\ell-m} + k, c_{\ell-m} + k + u)$ has been assigned to a node in R, then let c_i be the first such channel, and let w be the node to which this channel is assigned. Let A be the set containing all nodes that receive channels $\{c_{\ell-m}, \dots, c_{i-1}\}$.

Take tile $t' = \chi^A \in \mathcal{T}_{QR}$. Let $n_1 = |A \cap Q|$ and $m_1 = |A \cap R|$. Then $c_i \geq c_{\ell-m} + n_1 u + m_1 a + u - a$, so $c_i - c_{\ell-m} \geq c(t')$ By induction, there exists a tiling y_{end} that covers channels $\{c_i, \dots, c_f\}$ and has cost at most $c_f - c_i$. Let $y = y_R - \{p_R\} + \{t'\} + y_{end}$. Tiling y covers all channels, and has cost

$$c(y) = c(y_R - \{p_R\}) + c(t') + c(y_{end}) \leq c_f - c_0.$$

If no channel from the range $[c_{\ell-m} + k, c_{\ell-m} + k + u)$ is assigned to a node in Q, then let c_i be the first channel greater than or equal to $c_{\ell-m} + k + u$. If no such channel exist, take $i = f$. Let w be the node that channel c_i is assigned to. By induction, there exists a tiling y_{end} of cost at most $c_f - c_i$ that covers channels $\{c_i, \dots, c_f\}$. Let p be the patch of y_{end}.

Let A be the set of nodes that receive channels $\{c_{\ell-m}, \dots, c_{i-1}\}$, and let $n_1 = |Q \cap A|$, and $m_1 = |R \cap A|$. Since the set $\{c_{\ell-m}, \dots, c_i\}$ contains at least two crossovers, we have that $c_i \geq c_{\ell-m} + n_1 u + m_1 a + u - a$.

If $w \in R$, then make a tile $t' = \chi^A \in \mathcal{T}_{QR}$, which will have cost $c(t') = \max\{k, n_1 u + m_1 a + u - a\} \leq c_i - c_{\ell-m}$. Let $y = y_R - \{p_R\} + \{t'\} + y_{end}$. Then tiling y covers all channels, has a patch of the right type, and has cost $c(y) \leq c_f - c_0$.

If $w \in Q$, then it may be that p is in $\mathcal{P}_Q$ or $\mathcal{P}_{QR}^{big}$, which is not allowed. If $p \in \mathcal{P}_Q$, then add a node $v \in A_R$ to p, so let $p' = p + \chi^{\{v\}}$, of cost $c(p) + u$. Also, we have $t' = \chi^{A - \{v\}}$ of cost $\max\{k, n_1 u + (m_1 - 1)a + u - a\}$, if $m_1 > 1$, and cost $\max\{k, n_1 u\}$, if $m_1 = 1$. Since $w \in Q$, the channels $\{c_{\ell-m}, \dots, c_i\}$ contain at least three crossovers, so we have that $c_i \geq c_{\ell-m} + n_1 u + m_1 a + 2u - 2a$. Also, $c_i \geq c_{\ell-m} + k + u$, so $c(t') \leq c_i - c_{\ell-m} - u$

Let $y = y_R - \{p_R\} + \{p'\} + y_{end} - \{p\} + \{t'\}$. The tiling y covers all channels, has a patch of the right type, and has cost

$$\begin{aligned} c(y) &= c(y_R - \{p_R\}) + c(t') + c(y_{end}) - c(p) + c(p') \\ &\leq (c_{\ell-m} - c_0) + (c_i - c_{\ell-m} - u) + (c_f - c_i) + u \\ &= c_f - c_0. \end{aligned}$$

If $p \in \mathcal{P}^{big}_{QR}$, then let $V_Q = V(p) \cap Q$ and $V_R = V(p) \cap R$, and let B_Q and B_R be the sets of nodes of weight 2 in p from Q and R, respectively. Let $n^p = |V_Q|$, $n^p_2 = |B_Q|$, $m^p = |V^R|$ and $m^p_2 = |B^R|$. Then p has cost $(n^p + n^p_2)u + \max\{k, m^p a\} + (m^p_2 - 1)a$. Make a new tile $t' = \chi^{V^Q} + \chi^{V^R} \in \mathcal{T}_{QR}$, of cost $\max\{k, n^p u + m^p a + u - a\}$, another tile $t'' = \chi^A \in \mathcal{T}_{QR}$ of cost $\max\{k, n_1 u + m_1 a + u - a\}$, and a patch $p' = \chi^{B^Q} + \chi^{B^R} \in \mathcal{P}_{QR}$ of cost $n^p_2 u + (m^p_2 - 1)a$. So $c(t') + c(p') - c(p) = \max\{k, (n^p u + m^p a + u - a)\} + (n^p_2 u + (m^p_2 - 1)a) - ((n^p + n^p_2)u + \max\{k, m^p a\} + (m^p_2 - 1)a) \leq u - a$. Also,

Since $c_i \geq c_{\ell-m} + n_1 u + m_1 a + 2u - 2a$ and $c_i \geq c_{\ell-m} + u$, we also have that $c(t'') \leq c_i - c_{\ell-m} - u + a$.

Let $y = y_R - \{p_R\} + y_{end} - \{p\} + \{t'\} + \{t''\} + \{p'\}$. The tiling y covers all channels, has a patch of the right type, and has cost

$$\begin{aligned} c(y) &= c(y_R - \{p_R\}) + c(y_{end}) - c(p) + c(t') + c(p') + c(t'') \\ &\leq (c_{\ell-m} - c_0) + (c_f - c_i) + (u - a) + (c_i - c_{\ell-m} - u + a) \\ &= c_f - c_0. \end{aligned}$$

Case 2B. y_Q contains tiles.

By Lemma 5.2, patch p_Q covers channels $\{c_{j-n}, \ldots, c_{j-1}\}$, and these channels are all assigned to nodes in Q, so $j - n \geq \ell$. By induction there exists a tiling y_{end} that covers channels $\{c_{j-n}, \ldots, c_f\}$.

Let p be the patch of y_{end}, let $A_Q = V(p) \cap Q$ and $A_R = V(p) \cap R$, and let B_Q and B_R be the sets with nodes of weight 2 in Q and R, respectively. Let $n_1 = |A_Q|$, $m_1 = |A_R|$, $b_Q = |B_Q|$ and $b_R = |B_R|$. Let t be a tile from y_Q, and let V_Q be the support of t and $n_2 = |V_Q|$. Let $V_R = V(p_R)$.

In the following table, we show how we will combine p_R, p and t into a new tile t' and a new patch p'.

In all cases, we form the new tiling

$$y = y_R - \{p_R\} + y_Q - \{p_Q\} - \{t\} + y_{end} - \{p\} + \{t'\} + \{p'\}.$$

Case	Condition	Tile t'	Patch p'
(1)	$p \in \mathcal{P}_{QR}^{big}$	$p + \chi^X$ where $X \subseteq V_Q$, and $\lvert X\rvert = n_2 - n_1$	$t - \chi^X + \chi^{B_Q}$
(2)	$p \in \mathcal{P}_{QR}$		
(2.1)	(2) and $n_1u + m_1 + u - 1 \geq k$	p	p_R
(2.2)	(2) and $n_1u + m_1 + u - 1 < k$	$t + \chi^{A_R}$	$\chi^{A_Q} + p_R$
(3)	$p \in \mathcal{P}_Q$	t	$p_Q + p$

Case	Cost $c(t')$	$t' \in$	Cost $c(p')$	$p' \in$
(1)	$c(p) + c(t) - n_1u$	$\mathcal{T}_{QR}^{big}$	$n_1u + (m-1)a$	$\mathcal{P}_{QR}$
(2.1)	$n_1u + m_1a + u - a$	$\mathcal{T}_{QR}$	$c(p_R)$	$\mathcal{P}_R$
(2.2)	$n_2u + m_1a + u - a$	$\mathcal{T}_{QR}$	$n_1u + (m-1)a$	$\mathcal{P}_{QR}$
(3)	$c(t)$	$\mathcal{T}_Q$	$n_1u + (m_1 + m - 1)a$	$\mathcal{P}_{QR}$

Table 4: Combining patches

It can be verified, using the table, that in all cases, $c(p') + c(t') - c(t) - c(p_R) - c(p) \leq u$, so y has cost

$$\begin{aligned}
c(y) &= c(y_R) - c(p_R) + c(y_Q) - c(p_Q) + c(y_{end}) - c(t) - c(p) + c(t') + c(p') \\
&\leq (c_{\ell-1} - c_0) + (c_{j-1} - c_\ell) - (n-1)u + (c_f - c_0) + u \\
&\leq (c_{\ell-1} - c_0) + (c_{j-n} - c_\ell) + (c_f - c_{j-n}) + (c_\ell - c_{\ell-1}) \\
&\leq c_f - c_0
\end{aligned}$$

In each case, y covers all channels, the patch of y is of the right type, and it is straightforward to verify, using Tables 3 and 5, that $c(y) \leq c_f - c_0$. This completes the proof. □

6 Conclusions

We have described how a new general method of obtaining lower bounds for the channel assignment problems, when applied to the specific example of nested cliques, leads to a complete family of lower bounds which include almost all known clique-bounds, are easy to compute, and give improved results when applied to an example from the literature.

Futher work should address the computational issues related to lower bounds. A computational study comparing the performance of tile cover bounds to the TSP bound and the network flow bound on a number of realistic CAP instances would be a valuable addition to this theoretical analysis.

It is also an interesting question whether the tile cover approach can be used to obtain good channel assignments. Knowledge about which lower bound is most restrictive for any particular instance could be used to determine which tiles were most suited to build the best assignment.

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