Math 2505 - Explicit enumeration of $\mathbb{Q}$

The purpose of this note is to construct an explicit enumeration of the rational numbers. That is, we want to find a sequence $(r_n)_{n=1}^{\infty}$ such that

- $r_n \in \mathbb{Q}$, for all $n \in \mathbb{N}$.
- For every $s \in \mathbb{Q}$, there exists $n \in \mathbb{N}$ with $r_n = s$.
- If $n, m \in \mathbb{N}, n \neq m$, then $r_n \neq r_m$.

Thus, $n \to r_n$ is a one-to-one and onto mapping of $\mathbb{N}$ to $\mathbb{Q}$. A one-to-one and onto mapping is called a *bijection*.

Our first step is to construct a bijection between $\mathbb{N}_0$ and $\mathbb{Z}$. Note that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Recall that, for $x > 0$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Define $\rho : \mathbb{N}_0 \to \mathbb{Z}$ by

$$\rho(n) = (-1)^{n+1} \left\lfloor \frac{n+1}{2} \right\rfloor, \qquad \forall\, n \in \mathbb{N}_0. \tag{$*$}$$

So, $\rho(0) = 0, \rho(1) = 1, \rho(2) = -1, \rho(3) = 2, \rho(4) = -2, \rho(5) = 3, \cdots$. Another way of defining ρ is to say $\rho(0) = 0$, $\rho(2k) = -k$, and $\rho(2k-1) = k$, for any $k \in \mathbb{N}$. It is clear that ρ is a one-to-one function from $\mathbb{N}_0$ onto $\mathbb{Z}$.

DEFINITION: For $k, n \in \mathbb{N}$, we say k is a *factor* of n if $n/k \in \mathbb{N}$. We say n is a *prime number* if $n > 1$ and the only factors of n are 1 and n, itself.

The prime numbers are like the multiplicative atoms of the natural numbers. It is easy to show that any natural number can be expressed as a product of prime numbers in a manner that is unique. We want to formulate this statement carefully. Let $\mathcal{P}$ denote the set of all prime numbers.

PROPOSITION A: [Euclid] $\mathcal{P}$ is an infinite set.

PROOF: Suppose, to the contrary, that $\mathcal{P}$ has only finitely many members. That is, we can list the members of $\mathcal{P}$ as $p_1, p_2, \cdots, p_k$, for some $k \in \mathbb{N}$. Let $N = 1 + p_1 \cdot p_2 \cdot p_3 \cdots p_k$, one plus the product of these k prime numbers. Let q be a prime factor of N, so $N = qM$, for some $M \in \mathbb{N}$. If $q = p_j$, for some $1 \leq j \leq k$, then $L = p_1 \cdot p_2 \cdot p_3 \cdots p_k / q \in \mathbb{N}$. But this means that

$$1 = N - p_1 \cdot p_2 \cdot p_3 \cdots p_k = qM - qL = q(M - L)$$

and $M - L$ is an integer. This is a contradiction! Thus, $\mathcal{P}$ is an infinite set. $\square$

We can easily identify the first few members of $\mathcal{P}$. They are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc. using induction, we can define $p_j \in \mathbb{N}$, for each $j \in \mathbb{N}$, so that

- $p_j \in \mathcal{P}$, for all $j \in \mathbb{N}$.
- $p_j < p_{j+1}$, for all $j \in \mathbb{N}$.
- $\mathcal{P} = \{p_j : j \in \mathbb{N}\}$.

So $p_1, p_2, p_3, \cdots$ is a list of all the prime numbers in increasing order.

PROPOSITION B: For any $N \in \mathbb{N}, N > 1$, there exists a unique $k \in \mathbb{N}$ and unique $a_j \in \mathbb{N}_0$, $1 \leq j \leq k$, so that $a_k \geq 1$ and

$$N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}. \qquad (**)$$

PROOF: Fix $N \in \mathbb{N}$. There are only finitely many prime numbers less than or equal to N. So there are only finitely many prime factors of N. Let p be the largest of these. Then $p = p_k$, for some $k \in \mathbb{N}$. Let a_k denote the maximum natural number a so that p_k^a is a factor of N. That is $p_k^{a_k}$ is a factor of N and $p_k^{a_k+1}$ is not a factor of N. For $1 \leq j < k$, let a_j be the maximum number in $\mathbb{N}_0$ so that $p_j^{a_j}$ is a factor of N. Then $(**)$ holds with these values. $\square$

Some examples; $2 = 2^1$, $3 = 2^0 3^1$, $4 = 2^2$, $5 = 2^0 3^0 5^1$, $6 = 2^1 3^1$, etc. Any natural number larger than 1 is uniquely expressible as in Proposition B. Another example is $2548 = 2^2 3^0 5^0 7^2 11^0 13^1$. Play with finding the $(**)$ form for some integers between 100 and 10,000 to get a feel for this unique representation method.

PROPOSITION C: For any $r \in \mathbb{Q}, r > 0, r \neq 1$, there exists a unique $k \in \mathbb{N}$ and unique $b_j \in \mathbb{Z}$, $1 \leq j \leq k$ so that $b_k \neq 0$ and

$$r = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}. \qquad (***)$$

PROOF: Simply write $r = N/M$, where $N, M \in \mathbb{N}$ and have no factors in common other than 1. Then Proposition B is used to get the representation $(***)$ for r. $\square$

We use the bijection ρ between $\mathbb{N}_0$ and $\mathbb{Z}$, defined in $(*)$, to distribute the exponents of the primes and create a map between the natural numbers and the positive rationals. Define $\Phi : \mathbb{N} \to \mathbb{Q}^+$ by $\Phi(1) = 1$ and, if $N \in \mathbb{N}$, $N > 1$, is written as in $(**)$, then

$$\Phi(N) = \Phi(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = p_1^{\rho(a_1)} p_2^{\rho(a_2)} \cdots p_k^{\rho(a_k)}.$$

For example, $\Phi(2548) = \Phi\left(2^2 3^0 5^0 7^2 11^0 13^1\right) = 2^{-1} 3^0 5^0 7^{-1} 11^0 13^1 = \frac{13}{14}$.

It is easy and natural to extend this to a map Ψ of $\mathbb{Z}$ onto $\mathbb{Q}$ by reflection. That is, for $M \in \mathbb{Z}$,

$$\Psi(M) = \begin{cases} -\Phi(-M) & \text{if } M < 0 \\ 0 & \text{if } M = 0 \\ \Phi(M) & \text{if } M > 0. \end{cases}$$

PROPOSITION D: The function Ψ is one-to-one and onto from $\mathbb{Z} \to \mathbb{Q}^+$.

PROOF: This is just a simple matter of checking the cases. $\square$

Noting that $n \to \rho(n-1)$ is a one-to-one function $\mathbb{N}$ onto $\mathbb{Z}$, we can now define an explicit enumeration of the rational numbers. Define

$$r_n = \Psi(\rho(n-1)), \qquad \forall\, n \in \mathbb{N}.$$

Then $\mathbb{Q} = \{r_n \,:\, n \in \mathbb{N}\}$. This enumeration of the rational numbers (in bold) starts out as follows:

$r_1 = \Psi(0) = \mathbf{0}$ $\quad r_2 = \Psi(1) = \mathbf{1}$ $\quad r_3 = \Psi(-1) = -\mathbf{1}$ $\quad r_4 = \Psi(2) = \mathbf{2}$
$r_5 = \Psi(-2) = -\mathbf{2}$ $\quad r_6 = \Psi(3) = \mathbf{3}$ $\quad r_7 = \Psi(-3) = -\mathbf{3}$ $\quad r_8 = \Psi(4) = \mathbf{\frac{1}{2}}$
$r_9 = \Psi(-4) = -\mathbf{\frac{1}{2}}$ $\quad r_{10} = \Psi(5) = \mathbf{5}$ $\quad r_{11} = \Psi(-5) = -\mathbf{5}$ $\quad r_{12} = \Psi(6) = \mathbf{6}$

It might seem that we will never get through all the rational numbers at this rate. However, because of the explicit nature of the enumeration, we can easily find, for a given $r \in \mathbb{Q}$, the $n \in \mathbb{N}$ such that $r_n = n$. For example, consider $\frac{585}{392}$. For what $n \in \mathbb{N}$ is $r_n = \frac{585}{392}$? To answer this, we first factor the numerator and denominator into powers of prime numbers:

$$\frac{585}{392} = \frac{2^0 3^2 5^1 7^0 11^0 13^1}{2^3 3^0 5^0 7^2} = 2^{-3} 3^2 5^1 7^{-2} 11^0 13^1.$$

We note that $\rho(6) = -3, \rho(3) = 2, \rho(1) = 1$, and $\rho(4) = -2$. Thus,

$$\Psi(269,680,320) = \Psi(2^6 3^3 5^1 7^4 13^1) = 2^{-3} 3^2 5^1 7^{-2} 11^0 13^1 = \frac{585}{392}.$$

Now, if $n = 539,360,642$, then $\rho(n-1) = 269,680,320$. Thus,

$$r_{539,360,642} = \frac{585}{392}.$$

Try one yourself. Pick a fraction, either positive or negative, and figure out which r_n it is. It's fun!