Proof of the Fukui conjecture via resolution of singularities and related methods. IV*

(Short Title: Proof of the Fukui conjecture)

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* Dedicated to the memory of Prof. Kenichi Fukui (1918 ~ 1998).

Abstract

The present article is a direct continuation of the previous part III of this series of articles, which have been devoted to cultivating a new interdisciplinary region between chemistry and mathematics. In the present part IV, we develop two sets of fundamental theoretical tools, using methods from the field of resolution of singularities and analytic curves. These two sets of tools are essential in structurally elucidating the assertion of the Fukui conjecture (concerning the additivity problems) and the crux of the functional asymptotic linearity theorem (functional ALT) that proves the conjecture in a broad context. This conjecture is a vital guideline for a future development of the repeat theory (RST) – the central unifying theory in the First and the Second Generation Fukui Project.

Key words:

Fukui conjecture; repeat space theory (RST); additivity problems; Asymptotic Linearity Theorem (ALT); resolution of singularities

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1. Introduction

This article is a direct continuation of the previous part III [1] of this series of articles, which have been devoted to cultivating a new interdisciplinary region between chemistry and mathematics along the unifying spirit of the First and the Second Generation Fukui Project (cf. [1-12] and references therein).

In the previous part II of this series of articles [7], we have established the following sequence of logical implications

$$\text{LAP1} \Rightarrow \text{Functional ALT} \Rightarrow \text{the Fukui conjecture,}$$

where Local Analyticity Proposition version 1 (LAP1) was also referred to as the Target Proposition.

The Target Proposition (LAP1) is a major key to proving the Fukui conjecture via resolution of singularities and related methods. This proposition involves the union $\bigcup_{i=1}^{n} \Gamma(h_i)$ of special sets $\Gamma(h_i)$. Such a sum was called a multi-function-set in [1] and plays an important role in cross-disciplinary investigations using the repeat space theory (RST), which is the central theory in the First and the Second Generation Fukui Project. We remark that the energy band curves of carbon nanotubes given in article [3] are locally expressed in terms of multi-function-sets $\bigcup_{i=1}^{n} \Gamma(h_i)$.

To prove the Target Proposition via resolution of singularities and related methods, it is convenient to prepare four sets of fundamental tools:

(i) fundamental tools I (developed in part III of this series [1])

(ii) fundamental tools II (developed in part III of this series [1])

(iii) fundamental tools III (given in section 2 of the present part IV)

(iv) fundamental tools IV (given in section 3 of the present part IV)

The goal of the present part IV of this series of articles is to set up fundamental tools III and IV, and in particular, to establish Proposition 3.5.C in section 3, which studies the multi-function-set appearing in the above-mentioned Target Proposition. The Target Proposition will be proved in the forthcoming part V of this series using Proposition 3.5.C.

As in part III, in the present part IV of this series, we construct fundamental theoretical tools in such a way that they are also readily usable as modular parts of the repeat space theory (RST) [1-12] by which one can solve in a unifying manner a variety of molecular problems lying in the interdisciplinary region between chemistry and mathematics.

2. Fundamental tools III

Throughout, we retain the notation employed in the preceding parts I, II, and III of this series of articles [6,7,1]. (The reader is referred to refs. [1,6,7] for the notation.) Let $\mathbb{Z}^+$, $\mathbb{Z}_0^+$, $\mathbb{Z}$, $\mathbb{R}^+$, $\mathbb{R}_0^+$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively, the set of all positive integers, nonnegative integers, integers, positive real numbers, nonnegative real numbers, real numbers, complex numbers. Throughout this article, when a topological space $E = (X, o)$ is defined and when no confusion can arise, E also denotes its underlying set X.

Proposition 2.1. *Let $r \in \mathbb{R}^+$ and define topological subspace E_1 of $\mathbb{C}$ by*

$$E_1 = (\varDelta(r), o_1), \tag{2.1}$$

where o_1 denotes the relative topology on $\varDelta(r)$ induced by the usual Euclidean topology of $\mathbb{C}$. Let $f, g \in H(E_1)$. Define $\hat{u}_{f,g}: E_1 \to \mathbb{C}^2$ by

$$\hat{u}_{f,g}(t) = (f(t), g(t)). \tag{2.2}$$

Define topological subspace E_2 of $\mathbb{C}^2$ by

$$E_2 = (\hat{u}_{f,g}(E_1), o_2), \tag{2.3}$$

where o_2 denotes the relative topology on $\hat{u}_{f,g}(\varDelta(r))$ induced by the usual Euclidean topology of $\mathbb{C}^2$. Define $u_{f,g}: E_1 \to E_2$ by

$$u_{f,g}(t) = \hat{u}_{f,g}(t). \tag{2.4}$$

Let $m \in \mathbb{Z}^+$, and suppose that

$$f(t) = t^m \tag{2.5}$$

for all $t \in \varDelta(r)$ and that

$$g(0) = 0. \tag{2.6}$$

Let $q \in\ ]0, r[$. Then $u_{f,g}(\varDelta(q))$ is a neighborhood of the origin $(0, 0)$ of E_2.

Proof. The fact that $u(\varDelta(q))$ is a neighborhood of the origin $(0, 0)$ of E_2 follows from the propositions 2.2, 2.3, and 2.4 below. //

In what follows, we let

$$\hat{u} := \hat{u}_{f,g}, \tag{2.7}$$

$$u := u_{f,g}. \tag{2.8}$$

Proposition 2.2. *The notation and the assumption being as in proposition* 2.1, *we have*

$$u(\Delta(q)) \supset u(\Delta(r)) \cap \overline{f(\Delta(r) - \Delta(q))}^C \times \mathbb{C}, \tag{2.9}$$

where $^{-}$ *and* ^{C} *denote, respectively, the closure operation and the complement in* $\mathbb{C}$.

Proof. Let

$$(\alpha, \beta) \in u(\Delta(r)) \cap \overline{f(\Delta(r) - \Delta(q))}^C \times \mathbb{C}. \tag{2.10}$$

Then, there exists a $t_0 \in \Delta(r)$ such that

$$\alpha = f(t_0), \tag{2.11}$$

$$\beta = g(t_0), \tag{2.12}$$

and such that

$$f(t_0) \in \overline{f(\Delta(r) - \Delta(q))}^C. \tag{2.13}$$

Since $t_0 \in \Delta(r)$, we have either

$$t_0 \in \Delta(r) - \Delta(q), \tag{2.14}$$

or

$$t_0 \in \Delta(q). \tag{2.15}$$

But, in case (2.14),

$$f(t_0) \in f(\Delta(r) - \Delta(q)) \subset \overline{f(\Delta(r) - \Delta(q))}, \tag{2.16}$$

which contradicts (2.13), thus we have (2.15).

Hence, there exists a $t_0 \in \Delta(q)$ such that

$$\alpha = f(t_0), \tag{2.17}$$

$$\beta = g(t_0). \tag{2.18}$$

Therefore,

$$(\alpha, \beta) \in u(\Delta(q)). \tag{2.19}$$ //

Proposition 2.3. *The notation and the assumption being as in proposition* 2.1, *the set* $u(\Delta(r)) \cap \overline{f(\Delta(r)-\Delta(q))}^C \times \mathbb{C}$ *is an open set in* E_2.

Proof. Note that both $\overline{f(\Delta(r)-\Delta(q))}^C$ and $\mathbb{C}$ are open sets in $\mathbb{C}$, and hence $\overline{f(\Delta(r)-\Delta(q))}^C \times \mathbb{C}$ is open in $\mathbb{C}^2$. The conclusion follows. //

Proposition 2.4. *The notation and the assumption being as in proposition* 2.1, *we have*

$$(0, 0) \in u(\Delta(r)) \cap \overline{f(\Delta(r)-\Delta(q))}^C \times \mathbb{C}. \tag{2.20}$$

Proof. Since $u(0) = (0, 0)$, clearly $(0, 0) \in u(\Delta(r))$, hence it suffices to prove that

$$(0, 0) \in \overline{f(\Delta(r)-\Delta(q))}^C \times \mathbb{C}, \tag{2.21}$$

i.e.,

$$0 \notin \overline{f(\Delta(r)-\Delta(q))}. \tag{2.22}$$

Let F: $\overline{\Delta(r)} \to \mathbb{C}$ be such that

$$F(t) = t^m \tag{2.23}$$

for all $t \in \overline{\Delta(r)}$. Observe that

$$\overline{f(\Delta(r)-\Delta(q))} = \overline{F(\Delta(r)-\Delta(q))} \subset \overline{F(\overline{\Delta(r)}-\Delta(q))}, \tag{2.24}$$

and note that $\overline{\Delta(r)}-\Delta(q)$ is compact in $\overline{\Delta(r)}$ hence that $F(\overline{\Delta(r)}-\Delta(q))$ is compact in $\mathbb{C}$ because F is continuous.

By the definition of F, we have

$$0 \notin F(\overline{\Delta(r)}-\Delta(q)). \tag{2.25}$$

On the other hand, recalling the fact that any compact set in $\mathbb{C}$ is closed, we have

$$F(\overline{\Delta(r)} - \Delta(q)) = \overline{F(\overline{\Delta(r)} - \Delta(q))}. \tag{2.26}$$

Hence,

$$0 \notin \overline{F(\overline{\Delta(r)} - \Delta(q))}, \tag{2.27}$$

implying that (2.22) is true. //

3. Fundamental tools IV

We begin this section by introducing some new notation. For the fundamental properties of the UFD $\mathbb{C}\{z_1, z_2\}$ and other related UFDs, as well as for the Weierstrass Preparation Theorem, the reader is referred to refs. [13-15] and references therein.

Notation 3.1.

For $x \in \mathbb{C}$ and $r \in \mathbb{R}^+$, let

$$\Delta_x(r) := \{y \in \mathbb{C}: |y - x| < r\}.$$

For $x \in \mathbb{C}$, let

$$\Delta_x(\infty) := \mathbb{C}.$$

For $r \in \mathbb{R}^+$, let

$$\Delta(r) := \Delta_0(r).$$

For $\boldsymbol{x} = (x_1, x_2) \in \mathbb{C}^2$ and $\boldsymbol{r} = (r_1, r_2) \in \mathbb{R}^{+2}$, let

$$\Delta_{\boldsymbol{x}}(\boldsymbol{r}) := \Delta_{x_1}(r_1) \times \Delta_{x_2}(r_2).$$

For $\boldsymbol{r} = (r_1, r_2) \in \mathbb{R}^{+2}$, let

$$\Delta(\boldsymbol{r}) := \Delta(r_1) \times \Delta(r_2).$$

$\mathbb{C}\{z_1, z_2\}$: the UFD of convergent power series in the indeterminates z_1 and z_2.

$\nabla(\psi) := \{r \in \mathbb{R}^{+2}: \psi(x_1, x_2)$ is convergent as a series of complex numbers for all $(x_1, x_2) \in \Delta(r)\}$,

where $\psi \in \mathbb{C}\{z_1, z_2\}$.

$$\Pi(\mathbb{C}\{z_1, z_2\}) := \{\psi \in \mathbb{C}\{z_1, z_2\}: \psi(0, 0) = 0, \psi(0, \lambda) \not\equiv 0\}.$$

The following proposition and the outline of the proof played a significant role in preparing fundamental tools IV. Special thanks are due to

Prof. Isao Naruki (former member of the RIMS, Kyoto University and Ritsumeikan University, Japan) who provided the outline of the proof.

Proposition 3.1. *Let* $\mathbb{C}\{z_1, z_2\}$ *denote the unique factorization domain of convergent power series in* z_1 *and* z_2. *Let* ψ *be an irreducible element of* $\mathbb{C}\{z_1, z_2\}$ *with* $\psi(0, 0) = 0$ *and* $\psi(0, \lambda) \not\equiv 0$. *Then, there exist*

(i) *a neighborhood V of the origin* (0, 0),

(ii) *a positive integer k,*

(iii) *a sequence of complex numbers* $a_1, a_2, \ldots,$

such that the set of zeros (θ, λ) *of* ψ *in V is described by the Puiseux expansion of* λ:

$$\lambda(\theta) = \sum_{n=1}^{\infty} a_n \theta^{n/k}. \tag{3.1}$$

Outline of Proof: Any analytic curve can be desingularized by a finite succession of blowing-ups, hence there exists a local parametrization for each irreducible branch of an analytic curve. Given a local parametrization of an irreducible branch by

$$\theta = ct^k(1 + c_1 t + c_2 t^2 + \ldots), \tag{3.2}$$

$$\lambda = d_1 t + d_2 t^2 + \ldots, \tag{3.3}$$

set

$$u = u(t) = t(1 + c_1 t + c_2 t^2 + \ldots)^{1/k}. \tag{3.4}$$

Then $u = u(t)$ has an analytic inverse function in some neighborhood of the origin:

$$t = t(u) = \sum_{n=1}^{\infty} e_n (c^{-1/k} \theta^{1/k})^n. \tag{3.5}$$

On the other hand, λ is a power series of t; we consequently see that λ can be expanded in a Puiseux series of θ in a neighborhood of the origin (by supplementing topological arguments). //

Before proceeding further, the reader is asked to briefly review

(i) the following notation of Arg x and $\hat{\nu}_{k,j}$ (cf. [1]),

(ii) propositions 3.1 and 3.2 (in section 3. Fundamental Tools I) in [1]: the previous part III of this series.

Notation 3.2.

For $x \in \mathbb{C} - \{0\}$, let Arg x denote the unique real number $\theta \in [0, 2\pi[$ such that

$$x = |x| \exp(i\theta).$$

For $k \in \mathbb{Z}^+$ and $j \in \{1, ..., k\}$, define $\hat{\nu}_{k,j}$: $\mathbb{C} \to \mathbb{C}$ by

$$\hat{\nu}_{k,j}(x) = \begin{cases} 0 & \text{if } x = 0, \\ |x|^{1/k}\exp(i(\text{Arg } x + 2\pi j)/k) & \text{if } x \neq 0. \end{cases}$$

Proposition 3.2. *Let* $r \in \mathbb{R}^+$ *and let* $f, g \in H(\Delta(r))$ *be such that* $f(0) = g(0) = 0$. *Suppose that* $f(t) \not\equiv 0$, *and let* $k \in \mathbb{Z}^+$ *be the order of the multiplicity of the zero of* f *at the origin of* $\mathbb{C}$. *Then there exist an* $r_0 \in\ ]0, r]$, *an* $s \in \mathbb{R}^+$, *and an* $h \in H_0(\Delta(s^{1/k}))$ *with* $h(0) = 0$ *such that the following equality holds:*

$$\{(f(t), g(t)) \colon t \in \Delta(r_0)\} \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{j=1}^{k} \Gamma(h \circ \nu_j), \tag{3.6}$$

where ν_j: $\Delta(s) \to \Delta(s^{1/k})$ *is the function defined by* $\nu_j(x) = \hat{\nu}_{k,j}(x)$.

Proof. We shall prove this proposition in the setting of proposition 3.1[1]. Let f, $g \in H(\Delta(r))$ be such that $f(0) = g(0) = 0$ and $f(t) \not\equiv 0$, let k, μ, and s be as in proposition 3.1[1]. Then, proposition 3.2[1] implies that the following equality holds:

$$\{(f(t), g(t)) \colon t \in \Delta(r_0)\} \cap (\Delta(s) \times \mathbb{C})$$
$$= \bigcup_{j=1}^{k} \Gamma(g \circ i_2 \circ \mu \circ i_1 \circ \nu_j), \qquad (3.7)$$

where $i_1 \colon \Delta(s^{1/k}) \to \hat{h}_1(\Delta(r_0))$ denotes the inclusion mapping, $\mu \colon \hat{h}_1(\Delta(r_0)) \to \Delta(r_0)$ is analytic, $i_2 \colon \Delta(r_0) \to \Delta(r)$ denotes the inclusion mapping, and $g \colon \Delta(r) \to \mathbb{C}$ is analytic by the hypothesis.

Thus, the function $g \circ i_2 \circ \mu \circ i_1 \colon \Delta(s^{1/k}) \to \mathbb{C}$ is analytic on its domain, hence is analytic at the origin 0 in $\Delta(s^{1/k})$. Now it is easily seen that there exists an $s \in \mathbb{R}^+$ and an $h \in H_0(\Delta(s^{1/k}))$ such that

$$\{(f(t), g(t)) \colon t \in \Delta(r_0)\} \cap (\Delta(s) \times \mathbb{C})$$
$$= \bigcup_{j=1}^{k} \Gamma(h \circ \nu_j). \qquad (3.8)$$

Recall the definition of ν_j, and notice that $\nu_j(0) = 0$ for all $j \in \{1, ..., k\}$. By taking the intersection of each side of (3.8) and $\{0\} \times \mathbb{C}$, we then notice that

$$\{(f(t), g(t)) \colon t \in \Delta(r_0)\} \cap (\{0\} \times \mathbb{C})$$
$$= \{(0, h(0))\}, \qquad (3.9)$$

which implies that $h(0) = 0$. //

Remark. If k is not equal to 1, for each j, the mapping ν_j is not continuous entirely on its domain, but this does not affect our argument in the present paper.

Proposition 3.3.A. *Let* $\psi \in \Pi(\mathbb{C}\{z_1, z_2\})$ *be an irreducible element of the UFD* $\mathbb{C}\{z_1, z_2\}$, *let* $r = (r_1, r_2) \in \nabla(\psi)$, *and let*

$$W := \{(\theta, \lambda) \in \Delta(r): \psi(\theta, \lambda) = 0\}. \tag{3.10}$$

Then, there exist

(i) *a neighborhood* V *of the origin* $(0, 0)$ *in* $\mathbb{C}^2$,

(ii) $r_0 \in \mathbb{R}^+$,

(iii) $f, g \in H(\Delta(r_0))$ *with* $f(0) = g(0) = 0$,

such that

$$W \cap V = \{(f(t), g(t)): t \in \Delta(r_0)\} \tag{3.11}$$

and such that the mapping $u: \Delta(r_0) \to W \cap V$ *defined by* $u(t) = (f(t), g(t))$ *is a bijection that sends any neighborhood of the origin* 0 *in* $\Delta(r_0)$ *to a neighborhood of the origin* $(0, 0)$ *in* $W \cap V$.

We prove this proposition at the end of this section.

Proposition 3.3.B. *Let* $\psi \in \Pi(\mathbb{C}\{z_1, z_2\})$ *be an irreducible element of the UFD* $\mathbb{C}\{z_1, z_2\}$, *let* $r = (r_1, r_2) \in \nabla(\psi)$, *and let*

$$W := \{(\theta, \lambda) \in \Delta(r): \psi(\theta, \lambda) = 0\}. \tag{3.12}$$

Then, there exist

(i) *a neighborhood* V *of the origin* $(0, 0)$ *in* $\mathbb{C}^2$,

(ii) $r_0 \in \mathbb{R}^+$,

(iii) $f, g \in H(\Delta(r))$ *with* $f(0) = g(0) = 0$,

such that

$$W \cap V = \{(f(t), g(t)): t \in \Delta(r_0)\} \tag{3.13}$$

and such that the mapping $u: \Delta(r_0) \to W \cap V$ *defined by* $u(t) = (f(t), g(t))$ *is a homeomorphism.*

Proposition 3.4. *Proposition* 3.3.A *and proposition* 3.3.B *are equivalent.*

Proof. Proposition 3.3.B $\Rightarrow$ proposition 3.3.A is evident. We prove that proposition 3.3.A $\Rightarrow$ proposition 3.3.B:

Suppose that proposition 3.3.A is true, and let

(i) a neighborhood V_1 of the origin (0, 0) in $\mathbb{C}^2$,

(ii) $s \in \mathbb{R}^+$,

(iii) $f_1, g_1 \in H(\Delta(s))$ with $f_1(0) = g_1(0) = 0$,

be such that

$$W \cap V_1 = \{(f_1(t), g_1(t)) : t \in \Delta(s)\} \tag{3.14}$$

and such that the mapping $u_1: \Delta(s) \to W \cap V_1$ defined by $u_1(t) = (f_1(t), g_1(t))$ is a bijection that sends any neighborhood of the origin 0 in $\Delta(s)$ to a neighborhood of the origin (0, 0) in $W \cap V_1$.

Note first that $\Delta(s/2)$ is a neighborhood of the origin 0 in $\Delta(s)$, and hence that $u_1(\Delta(s/2))$ is a neighborhood of the origin (0, 0) in $W \cap V_1$. This implies that there exists a neighborhood V_2 of the origin (0, 0) in $\mathbb{C}^2$ such that

$$W \cap V_1 \cap V_2 = \{(f_1(t), g_1(t)) : t \in \Delta(s/2)\} = u_1(\Delta(s/2)). \tag{3.15}$$

Set

$$V = V_1 \cap V_2, \tag{3.16}$$

then V is obviously a neighborhood the origin (0, 0) in $\mathbb{C}^2$. Set

$$r_0 = s/2, \tag{3.17}$$

$$f = f_1 \mid \Delta(r_0), \tag{3.18}$$

$$g = g_1 \mid \Delta(r_0), \tag{3.19}$$

and define $u: \Delta(r_0) \to u_1(\Delta(r_0))$ by

$$u(t) = (f(t), g(t)), \tag{3.20}$$

or equivalently by

$$u(t) = u_1(t). \tag{3.21}$$

To verify that proposition 3.3.A $\Rightarrow$ proposition 3.3.B, it now suffices to prove that u is a homeomorphism.

Note that $\overline{\Delta(r_0)}$ is compact and that the mapping h: $\overline{\Delta(r_0)} \to u_1(\overline{\Delta(r_0)})$ defined by

$$h(t) = u_1(t) \tag{3.22}$$

is a continuous bijection and hence homeomorphism by the following well-known fact: If h is a continuous bijection from a compact topological space to a Hausdorff space, then h is a homeomorphism.

If h' is a homeomorphism of a topological space X onto Y and if X_0 is a nonempty subset of X, then u': $X_0 \to h'(X_0)$ defined by $u'(t) = h'(t)$ is a homeomorphism. Applying this argument to h and u, we see that u is a homeomorphism. //

The following proposition can be proved by using either proposition 3.3.A or 3.3.B; we shall use the former.

Proposition 3.5.A. *Let* $\psi \in \Pi(\mathbb{C}\{z_1, z_2\})$ *be an irreducible element of the UFD* $\mathbb{C}\{z_1, z_2\}$*, let* $r = (r_1, r_2) \in \nabla(\psi)$*, and let*

$$W := \{(\theta, \lambda) \in \Delta(r)\colon \psi(\theta, \lambda) = 0\}. \tag{3.23}$$

Then, there exist

(i) *a neighborhood* V *of the origin* $(0, 0)$ *in* $\mathbb{C}^2$,

(ii) $s \in \mathbb{R}^+$,

(iii) $k \in \mathbb{Z}^+$,

(iv) $h \in H_0(\Delta(s^{1/k}))$ *with* $h(0) = 0$,

such that the following equality holds:

$$W \cap V \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{j=1}^{k} \Gamma(h \circ \nu_j), \tag{3.24}$$

where ν_j: $\Delta(s) \to \Delta(s^{1/k})$ *is the function defined by* $\nu_j(x) = \hat{\nu}_{k,j}(x)$.

Proof. Under the assumptions of the proposition, by using proposition 3.3.A, we see that there exist

(i) a neighborhood V_1 of the origin (0, 0) in $\mathbb{C}^2$,

(ii) $r_0 \in \mathbb{R}^+$,

(iii) $f, g \in H(\Delta(r_0))$ with $f(0) = g(0) = 0$,

such that

$$W \cap V_1 = \{(f(t), g(t)) \colon t \in \Delta(r_0)\} \tag{3.25}$$

and such that the mapping $u \colon \Delta(r_0) \to W \cap V_1$ defined by $u(t) = (f(t), g(t))$ is a bijection that sends any neighborhood of the origin 0 in $\Delta(r_0)$ to a neighborhood of the origin (0, 0) in $W \cap V_1$.

We claim that $f(t) \not\equiv 0$. Suppose the contrary: $f(t) \equiv 0$. Then the hypothesis $\psi(0, \lambda) \not\equiv 0$ implies that $g \in H(\Delta(r_0))$ is not an open mapping, hence that g is a constant function. Since $g(0) = 0$, we have $g(t) \equiv 0$. But, then we would have

$$W \cap V_1 = \{(0, 0)\}, \tag{3.26}$$

which is impossible in view of the Weierstrass Preparation Theorem [14,15]. Hence, our claim is true. Let $k \in \mathbb{Z}^+$ be the order of the multiplicity of the zero of f at the origin in $\mathbb{C}$.

Now we may apply proposition 3.2, and we see that there exist an $r_{00} \in\ ]0, r_0]$, an $s \in \mathbb{R}^+$, and an $h \in H_0(\Delta(s^{1/k}))$ with $h(0) = 0$ and such that the following equality holds:

$$\{(f(t), g(t)) \colon t \in \Delta(r_{00})\} \cap (\Delta(s) \times \mathbb{C})$$
$$= \bigcup_{j=1}^{k} \Gamma(h \circ \nu_j). \tag{3.27}$$

On the other hand, since $\Delta(r_{00})$ is a neighborhood of the origin 0 in $\Delta(r_0)$, we see that $u(\Delta(r_{00})) = \{(f(t), g(t)) \colon t \in \Delta(r_{00})\}$ is a neighborhood of the origin (0, 0) in $W \cap V_1$. This implies that there exists a neighborhood V_2 of the origin in $\mathbb{C}^2$ such that

$$\{(f(t), g(t)): t \in \Delta(r_{00})\} = W \cap V_1 \cap V_2. \tag{3.28}$$

Set

$$V = V_1 \cap V_2, \tag{3.29}$$

then V is a neighborhood of the origin (0, 0) in $\mathbb{C}^2$. By combining (3.27), (3.28), and (3.29), we get the conclusion. //

Proposition 3.5.B. *Let* $\psi \in \Pi(\mathbb{C}\{z_1, z_2\})$ *be an irreducible element of the UFD* $\mathbb{C}\{z_1, z_2\}$, *let* $r = (r_1, r_2) \in \nabla(\psi)$, *and let*

$$W := \{(\theta, \lambda) \in \Delta(r): \psi(\theta, \lambda) = 0\}. \tag{3.30}$$

Then, there exist

(i) $s' \in \mathbb{R}^{+2}$,

(ii) $s \in \mathbb{R}^{+}$,

(iii) $k \in \mathbb{Z}^{+}$,

(iv) $h \in H_0(\Delta(s^{1/k}))$ *with* $h(0) = 0$,

such that the following equality holds:

$$W \cap \Delta(s') \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{j=1}^{k} \Gamma(h \circ \nu_j), \tag{3.31}$$

where $\nu_j: \Delta(s) \to \Delta(s^{1/k})$ *is the function defined by* $\nu_j(x) = \hat{\nu}_{k,j}(x)$.

Proposition 3.6. *Proposition* 3.5.A *and proposition* 3.5.B *are equivalent*.

Proof. Proposition 3.5.B $\Rightarrow$ proposition 3.5.A is evident. We prove that proposition 3.5.A $\Rightarrow$ proposition 3.5.B:

Suppose that proposition 3.5.A is true, and let

(i) a neighborhood V of the origin (0, 0) in $\mathbb{C}^2$,

(ii) $s_1 \in \mathbb{R}^+$,

(iii) $k \in \mathbb{Z}^+$,

(iv) $h \in H_0(\Delta(s_1{}^{1/k}))$ with $h(0) = 0$,

be such that

$$W \cap V \cap (\Delta(s_1) \times \mathbb{C}) = \bigcup_{j=1}^{k} \{(x, h(\nu_j(x)):\ x \in \Delta(s_1)\}. \tag{3.32}$$

Since V is a neighborhood of the origin of $\mathbb{C}^2$, for some $s' = (s'_1, s'_2) \in \mathbb{R}^{+2}$, one has

$$\Delta(s') \subset V. \tag{3.33}$$

Fix such an s', and also fix an $s \in\ ]0, s'_1]$ such that

$$s < s'_1, \tag{3.34}$$

and such that

$$|h(\nu_j(x)) - h(\nu_j(0))\ | = |h(\nu_j(x))| < s'_2 \tag{3.35}$$

for all $x \in \Delta(s)$ and for all $j \in \{1, ..., k\}$. (Note that all the $h \circ \nu_j$ are continuous at the origin of $\mathbb{C}$.)

Now it remains to prove that

$$W \cap \Delta(s') \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{j=1}^{k} \{(x, h(\nu_j(x)):\ x \in \Delta(s)\}. \tag{3.36}$$

By (3.34) and (3.35), we see that the relations

$$\Delta(s') \cap (\Delta(s) \times \mathbb{C}) = \Delta(s) \times \Delta(s'_2) \supset \bigcup_{j=1}^{k} \{(x, h(\nu_j(x)): x \in \Delta(s)\} \tag{3.37}$$

hold and hence that the relation

$$W \cap \Delta(s') \cap (\Delta(s) \times \mathbb{C}) \supset W \cap (\bigcup_{j=1}^{k} \{(x, h(\nu_j(x)): x \in \Delta(s)\}) \tag{3.38}$$

holds. But, (3.32) implies

$$W \supset \bigcup_{j=1}^{k} \{(x, h(\nu_j(x)): x \in \Delta(s)\} \tag{3.39}$$

holds. Thus, the right-hand side of (3.38) is equal to that of (3.36).

We now only have to show that

$$W \cap \Delta(s') \cap (\Delta(s) \times \mathbb{C}) \subset \bigcup_{j=1}^{k} \{(x, h(\nu_j(x)) : x \in \Delta(s)\}, \qquad (3.40)$$

but this relation is evidently true since

$$W \cap \Delta(s') \cap (\Delta(s) \times \mathbb{C})$$

$$\subset W \cap V \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{j=1}^{k} \{(x, h(\nu_j(x)) : x \in \Delta(s)\}. \qquad (3.41)$$

//

The second proof of proposition 3.1. The conclusion directly follows from propositions 3.5.A. //

Proof of proposition 3.3.A. We shall prove proposition 3.3.A by using the following proposition:

Proposition 3.7. *Let* $\psi \in \Pi(\mathbb{C}\{z_1, z_2\})$ *be an irreducible element of the UFD* $\mathbb{C}\{z_1, z_2\}$, *let* $r = (r_1, r_2) \in \nabla(\psi)$, *and let*

$$W := \{(\theta, \lambda) \in \Delta(r) : \psi(\theta, \lambda) = 0\}. \qquad (3.42)$$

Then, there exist

(i) *a neighborhood* V *of the origin* (0, 0) *in* $\mathbb{C}^2$,

(ii) $r_0 \in \mathbb{R}^+$,

(iii) $m \in \mathbb{Z}^+$,

(iv) $g \in H(\Delta(r_0))$ *with* $g(0) = 0$,

such that

$$W \cap V = \{(t^m, g(t)) : t \in \Delta(r_0)\} \qquad (3.43)$$

and such that the mapping $u: \Delta(r) \to W \cap V$ *defined by* $u(t) = (t^m, g(t))$ *is a bijection.*

Proof of proposition 3.7. This is an immediate consequence of theorem 5.3 and theorem 5.7 in [14]: Phillip A. Griffiths, Introduction to Algebraic Curves (Am. Math. Soc., Providence, 1989). //

In view of proposition 3.7, in order to prove proposition 3.3.A, it is enough to show that the above function u sends any neighborhood of the origin 0 in $\Delta(r)$ to a neighborhood of the origin (0, 0) in $W \cap V$.

But, by proposition 2.1 (in section 2. Fundamental Tools III), we know that for any $q \in (0, r)$, $u(\Delta(q))$ is a neighborhood of the origin (0, 0) in $W \cap V$, from which it immediately follows that the function u sends any neighborhood of the origin 0 in $\Delta(r)$ to a neighborhood of the origin (0, 0) in $W \cap V$. //

The following proposition is a general form of proposition 3.5.B and it easily follows from proposition 3.5.B.

Proposition 3.5.C. *Let* $\psi \in \Pi(\mathbb{C}\{z_1, z_2\})$, *let* $r = (r_1, r_2) \in \nabla(\psi)$, *and let*

$$W := \{(\theta, \lambda) \in \Delta(r) \colon \psi(\theta, \lambda) = 0\}. \tag{3.44}$$

Then, there exist

(i) $s' \in \mathbb{R}^{+2}$,

(ii) $s \in \mathbb{R}^{+}$,

(iii) $n \in \mathbb{Z}^{+}$, $k_1, \ldots, k_n \in \mathbb{Z}^{+}$,

(iv) $h_1 \in H_0(\Delta(s^{1/k_1})), \ldots, h_n \in H_0(\Delta(s^{1/k_n}))$ *with* $h_1(0) = \ldots = h_n(0) = 0$,

such that the following equality holds:

$$W \cap \Delta(s') \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \Pi(h_i \circ v_{i,j}), \tag{3.45}$$

where $v_{i,j}\colon \Delta(s) \to \Delta(s^{1/k_i})$ *is the function defined by* $v_{i,j}(x) = \hat{v}_{k_i, j}(x)$, *and where* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, k_i\}$.

Proof. Since $\mathbb{C}\{z_1, z_2\}$ is a UFD, ψ can be expressed as a finite product of irreducible elements of $\mathbb{C}\{z_1, z_2\}$. Bearing this fact in mind, we see that there exist $r' = (r'_1, r'_2) \in \mathbb{R}^{+2}$, $n \in \mathbb{Z}^+$, and irreducible elements $\psi_1, ..., \psi_n$ of $\mathbb{C}\{z_1, z_2\}$ which are all convergent for all $(\theta, \lambda) \in \Delta(r')$, such that

$$\psi(\theta, \lambda) = \psi_1(\theta, \lambda) \cdots \psi_n(\theta, \lambda) \tag{3.46}$$

for all $(\theta, \lambda) \in \Delta(r')$. Note that consequently the set of zeros of ψ in $\Delta(r')$ can be written as the union of the sets of zeros of ψ in $\Delta(r')$:

$$\begin{aligned}&\{(\theta, \lambda) \in \Delta(r'): \psi(\theta, \lambda) = 0\}\\ &= \{(\theta, \lambda) \in \Delta(r'): \bigvee_{i=1}^{n} [\psi_i(\theta, \lambda) = 0]\}\\ &= \bigcup_{i=1}^{n} \{(\theta, \lambda) \in \Delta(r'): \psi_i(\theta, \lambda) = 0\}.\end{aligned} \tag{3.47}$$

Now the conclusion easily follows from proposition 3.5.B. //

In the forthcoming part V of this series, proposition 3.5.C proved above plays a key role in establishing our Target Proposition (Local Analyticity Proposition version 1 (LAP1)).

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