DECOMPOSING THE WAVELET REPRESENTATION FOR SHIFTS BY WALLPAPER GROUPS

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Abstract. The wavelet group and wavelet representation associated with shifts coming from a two dimensional crystal symmetry group Γ and dilations by powers of 3, are defined and studied. The main result is an explicit decomposition of this 3Γ-wavelet representation into irreducible representations of the wavelet group. Because we prove that the 3Γ-wavelet representation is multiplicity free, this direct integral decomposition is essentially unique.

1. Introduction

In [17] and [10], the classical concept of wavelets on $\mathbb{R}^n$ was modified by replacing shifts by points from a lattice in $\mathbb{R}^n$ with "shifts" by the isometries from a crystal symmetry group, Γ. In this generalization, the dilation matrix A must be compatible with the action of Γ. For $\gamma \in \Gamma$, let $R(\gamma)$ denote the unitary shift operator on the Hilbert space $L^2(\mathbb{R}^n)$ defined by $R(\gamma)f(y) = f(\gamma^{-1} \cdot y)$, for all $y \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$. Here $\gamma^{-1} \cdot y$ denotes the result of shifting y by the isometry γ^{-1}. When Γ is the smallest crystal symmetry group consisting of translations by elements of $\mathbb{Z}^n$, then R gives the classical shifts. The dilation unitary, D_A, is given by $D_A f(y) = |\det(A)|^{1/2} f(Ay)$, for all $y \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$. The purpose of the present paper is to initiate a study of the operator algebraic and group representational structures of $\mathcal{G}(A, \Gamma)$, the smallest group of unitary operators containing both D_A and $R(\Gamma)$, in the case where Γ is a wallpaper group; that is, a symmetry group of a two dimensional crystal. We take the point of view, that $\mathcal{G}(A, \Gamma)$ is the image of a unitary representation of a particular group we define in section 2.

This paper partly generalizes [16], where such a study was carried out when the shifts were from the integer lattice, $\mathbb{Z}^n$. In that case, the corresponding representation was shown to be unitarily equivalent to an explicit direct integral of irreducible representations. The parameter space for the direct integral can be taken to be a wavelet set in the sense of [3] and [4]. In [20], [21], and [22], wavelet sets with very simple geometric structure (a finite union of convex sets) are constructed for increasingly more general dilation matrices. The abstract

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group being represented is not a Type I group. Thus, representations of this group need not have a unique direct integral decomposition into irreducible representations in general ([19]). So, it was interesting that its natural representation on $L^2(\mathbb{R}^n)$ could be shown to multiplicity free, and could be so nicely decomposed into irreducibles. See [2], [5], [6], and [7] for works partially motivated by the direct integral decomposition obtained in [16].

Although much of what we do below can be carried out in arbitrary dimensions, we restrict ourselves to the two dimensional case where computational details are more manageable. We also note that the analog of simple wavelet sets for shifts by a crystal symmetry group have recently been constructed, see [23], for all two dimensional crystal groups and an appropriate dilation matrix. Besides restricting to two dimensions, we further reduce notational details by using dilation by 3, which is compatible with all wallpaper groups, in all cases.

After introducing our notation and basic definitions in Section 2, we construct a particular semi-direct product group, denoted $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$, and at the end of Section 2 and in Section 3 we define the 3Γ-wavelet representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and show that this representation is multiplicity free, i.e. has an abelian commutant. It thus can be uniquely decomposed into a direct integral of irreducible representations. This is the unitary representation whose image is $\mathcal{G}(A,\Gamma)$, where the matrix A dilates by 3. In Section 4, we give an explicit construction of a family of irreducible unitary representations of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. In the final section, we display the 3Γ-wavelet representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ as the direct integral of the irreducible representations constructed in Section 4.

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2. Notation and Basic Results

We have selected notation to emphasize the central role of the natural unitary representation of the group of affine transformations of $\mathbb{R}^n$ and so that, as much as possible, we are consistent with the notation of [17] and [16].

Let $n \in \mathbb{N}$ and let $\mathcal{GL}_n(\mathbb{R})$ denote the group of invertible linear transformations of $\mathbb{R}^n$. Let id denote the identity in $\mathcal{GL}_n(\mathbb{R})$. For any $x \in \mathbb{R}^n$ and $L \in \mathcal{GL}_n(\mathbb{R})$, define the transformation $[x, L]$ of $\mathbb{R}^n$ by

$$[x, L]z = L(z + x), \quad \text{for all } z \in \mathbb{R}^n.$$

Let $\mathrm{Aff}(\mathbb{R}^n) = \{[x, L] : x \in \mathbb{R}^n, L \in \mathcal{GL}_n(\mathbb{R})\}$. Under composition as group product, $\mathrm{Aff}(\mathbb{R}^n)$ is the group of all invertible *affine transformations* of $\mathbb{R}^n$. Note that, for $[x, L], [y, M] \in \mathrm{Aff}(\mathbb{R}^n)$,

$$[x, L]\big([y, M]z\big) = L\big(M(z + y) + x\big) = LM\big(z + (M^{-1}x + y)\big) = [M^{-1}x + y, LM]z,$$

for all $z \in \mathbb{R}^n$. Thus, $[0, \mathrm{id}]$ is the identity in $\mathrm{Aff}(\mathbb{R}^n)$,

$$[x, L][y, M] = [M^{-1}x + y, LM], \quad \text{and} \quad [x, L]^{-1} = [-Lx, L^{-1}].$$

We note that the action of $\mathrm{Aff}(\mathbb{R}^n)$ on $\mathbb{R}^n$ provides a natural unitary representation of $\mathrm{Aff}(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$, which we will denote by R regardless of the dimension n. That is, for $[x, L] \in \mathrm{Aff}(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$,

$$R[x, L]g(y) = |\det(L)|^{-1/2} g\big([x, L]^{-1}y\big) = |\det(L)|^{-1/2} g(L^{-1}y - x), \quad \text{for all } y \in \mathbb{R}^n. \tag{1}$$

We will refer to R as the *natural representation*.

Let $\mathrm{Trans}(\mathbb{R}^n) = \{[x, \mathrm{id}] : x \in \mathbb{R}^n\}$, the normal subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ consisting of pure translations. Define $q : \mathrm{Aff}(\mathbb{R}^n) \to \mathcal{GL}_n(\mathbb{R})$ by $q[x, L] = L$, for all $[x, L] \in \mathrm{Aff}(\mathbb{R}^n)$. Then q is a homomorphism onto $\mathcal{GL}_n(\mathbb{R})$ with $\ker(q) = \mathrm{Trans}(\mathbb{R}^n)$. We note that the restriction of the natural representation R to $\mathrm{Trans}(\mathbb{R}^n)$ gives us the usual translation unitaries. That is, if $x \in \mathbb{R}^n$, then $R[x, \mathrm{id}] = T_x$, where $T_x f(y) = f(y - x)$, for all $y \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$.

Let $\mathcal{O}_n$ denote the group of orthogonal transformations of $\mathbb{R}^n$ and let $\mathrm{Iso}(\mathbb{R}^n)$ denote the subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ consisting of all affine transformations of the form $[x, L]$, with $x \in \mathbb{R}^n$ and $L \in \mathcal{O}_n$. These are the rigid motions of $\mathbb{R}^n$. An n-dimensional crystal group is a discrete subgroup Γ of $\mathrm{Iso}(\mathbb{R}^n)$ such that the quotient space $\mathbb{R}^n/\Gamma$ is compact. We refer to a 2-dimensional crystal group as *wallpaper group*. We will restrict our attention to wallpaper groups from now on.

Let Γ be a fixed wallpaper group. Let $\mathcal{N} = \mathrm{Trans}(\mathbb{R}^2) \cap \Gamma$, the pure translations in Γ. Then $\mathcal{N}$ is a normal subgroup of Γ. There are two linearly independent vectors $u, v \in \mathbb{R}^2$ such that $\mathcal{N} = \{[ju + kv, \mathrm{id}] : (j, k) \in \mathbb{Z}^2\}$. Therefore $\mathcal{N}$ is isomorphic to the lattice $N_\Gamma = \{ju + kv : (j, k) \in \mathbb{Z}^2\}$ in $\mathbb{R}^2$ and, hence, isomorphic to $\mathbb{Z}^2$. Let

$$\mathcal{D} = q(\Gamma) = \{L \in \mathcal{O}_2 : [x, L] \in \Gamma, \text{ for some } x \in \mathbb{R}^2\}.$$

Then $\mathcal{D}$ is a finite subgroup of $\mathcal{O}_2$ called the *point group* of Γ. There are 10 possibilities for $\mathcal{D}$ and they separate into two types depending on whether or not the group contains a reflection. If there is no reflection, then $\mathcal{D}$ is either the trivial group or generated by a rotation of order 2, 3, 4, or 6. Let $\rho(\theta)$ denote rotation through the angle θ. If $k \in \{1, 2, 3, 4, 6\}$, then the cyclic group of order k is $\mathcal{C}_k = \{\rho(2\pi/k)^j : 0 \le j \le k - 1\}$. Note that $\mathcal{C}_1$ is the trivial group. If $\mathcal{D}$ contains a reflection S about some one dimensional subspace of $\mathbb{R}^2$, then there is a $k \in \{1, 2, 3, 4, 6\}$ such that

$$\mathcal{D} = \mathcal{D}_k = \{\rho(2\pi/k)^j : 0 \leq j \leq k-1\} \cup \{\rho(2\pi/k)^j S : 0 \leq j \leq k-1\}.$$

For proofs of these standard facts about wallpaper groups, see [1],[8],[24].

If we denote the restriction of q to Γ by q again, then q is a homomorphism of Γ onto $\mathcal{D}$ and $\ker(q) = \mathcal{N}$. Thus $\mathcal{D}$ is isomorphic with $\Gamma/\mathcal{N}$. If $[0, L] \in \Gamma$, for all $L \in \mathcal{D}$, then Γ is called *symmorphic*. Otherwise, Γ is *nonsymmorphic* and there is no subgroup of Γ with the property that q restricted to that subgroup is an isomorphism with $\mathcal{D}$. In either case, we define

$$\mathcal{D}^0 = \{L \in \mathcal{O}_2 : [0, L] \in \Gamma\},$$

and note that Γ is symmorphic exactly when $\mathcal{D}^0 = \mathcal{D}$.

We draw particular attention to the wallpaper group denoted *pg*. This group is generated by $\{[u, \mathrm{id}], [v, \mathrm{id}], [(1/2)v, S]\}$, where $S \in \mathcal{O}_2$ is reflection in the line determined by multiples of v. The group *pg* has a rectangular lattice [24], so we can assume that u and v are along the horizontal and vertical axes respectively, so that $[(1/2)v, S]$ moves up by one half of a vertical translation unit and reflects about the vertical axis. This is called a *glide-reflection*. We have

$$pg = \{[ju + kv, \mathrm{id}] : (j, k) \in \mathbb{Z}^2\} \bigcup \left\{ \left[ju + \left(k + \frac{1}{2}\right) v, S \right] : (j, k) \in \mathbb{Z}^2 \right\}.$$

Note that *pg* is nonsymmorphic as $[0, S] \notin pg$, and that here we have $\mathcal{D} = \{\mathrm{id}, \mathrm{S}\}$, while $\mathcal{D}^0 = \{\mathrm{id}\}$.

For a general wallpaper group Γ, we will be interested in

$$T_\Gamma = \{x \in \mathbb{R}^2 : [x, L] \in \Gamma, \text{ for some } L \in \mathcal{D}\},$$

and also, for each $L \in \mathcal{D}$,

$$T_\Gamma^L = \{x \in \mathbb{R}^2 : [x, L] \in \Gamma\}.$$

If Γ is symmorphic, then $T_\Gamma = \{ju + kv : (j, k) \in \mathbb{Z}^2\}$, a lattice in $\mathbb{R}^2$. However, there are four nonsymmorphic wallpaper groups. They are *pg*, *pmg*2, *pgg*2, and *p*4*mg* in a standard notation scheme (see e.g. [24] or [27]). For each of these, extra elements are added to the lattice to form T_Γ, as described in the following lemma.

Lemma 2.1. *If Γ is a nonsymmorphic wallpaper group and $u, v \in \mathbb{R}^2$ are such that $\mathcal{N} = \{[ju + kv, \mathrm{id}] : (j, k) \in \mathbb{Z}^2\}$, then*

(1) *$u \perp v$, so we may assume $u = (1, 0)$ and $v = (0, 1)$.*

(2) *There exists a* $z \in \mathbb{R}^2$ *such that* $T_\Gamma^S = \left\{ju + kv + \frac{1}{2}z : (j,k) \in \mathbb{Z}^2\right\}$, *where* S *is reflection in the vertical axis. If* Γ *is either pg or pmg2, we can take* $z = v$*; if* Γ *is either pgg2 or p4mg, we can take* $z = u + v$.
(3) *For* $L \in \mathcal{D}$, $L \neq S$, $T_\Gamma^L = \{ju + kv : (j,k) \in \mathbb{Z}^2\}$

Proof. See, e.g.[24]. □

Let $A \in \mathcal{GL}_2(\mathbb{R})$. We say A is *compatible with* Γ if all eigenvalues of A have absolute value larger than 1 (that is, A is a dilation matrix), $[0,A]\Gamma[0,A]^{-1} \subseteq \Gamma$ (in which case $[0,A]\Gamma[0,A]^{-1}$ is a subgroup of Γ), and $\Gamma/[0,A]\Gamma[0,A]^{-1}$ is finite. Let $[x,L] \in \Gamma$. Then

$$[0,A][x,L][0,A]^{-1} = [x,AL][0,A^{-1}] = [Ax, ALA^{-1}].$$

By looking at the second component in the above calculation, we see that A compatible with Γ means $A\mathcal{D}A^{-1} \subseteq \mathcal{D}$. Thus, since conjugation by A is one-to-one and $\mathcal{D}$ is finite, $A\mathcal{D}A^{-1} = \mathcal{D}$. Taking $L = \mathrm{id}$ in the calculation gives that $AN_\Gamma \subseteq N_\Gamma \subseteq A^{-1}N_\Gamma$. Finally, since all eigenvalues of A have absolute value greater than 1, we have that $\cap_{k=1}^{\infty} A^k N_\Gamma = \{0\}$ and $\cup_{k=1}^{\infty} A^{-k} N_\Gamma$ is dense in $\mathbb{R}^2$.

Remark 2.2. *If* $A \in \mathcal{GL}_2(\mathbb{R})$ *is compatible with* Γ*, then applying the natural representation, given by* (1)*, to* $[0,A]$ *gives* $R[0,A]g(y) = |\det(A)|^{-1/2} g(A^{-1}y)$*, for* $y \in \mathbb{R}^2, g \in L^2(\mathbb{R}^2)$*. That is,* $R[0,A] = D_{A^{-1}}$ *in the notation of* [16]*. Recall, for* $x \in N_\Gamma$, $R[x,\mathrm{id}]g(y) = g(y-x)$*, for* $y \in \mathbb{R}^2, g \in L^2(\mathbb{R}^2)$*. Thus,* $R(\Gamma) \cup \{R[0,A^k] : k \in \mathbb{Z}\}$ *contains all the shifts by vectors in the lattice* N_Γ *and dilations by powers of* A*.*

Definition 2.3. If Γ is a wallpaper group and $A \in \mathcal{GL}_2(\mathbb{R})$ is compatible with Γ, let $\mathcal{G}(A,\Gamma)$ denote the smallest group of unitary operators on $L^2(\mathbb{R}^2)$ containing $R[0,A]$ and the set $R(\Gamma)$.

If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of bounded linear operators on $\mathcal{H}$. For $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, the *commutant* of $\mathcal{S}$ is $\mathcal{S}' = \{B \in \mathcal{B}(\mathcal{H}) : BS = SB, \forall S \in \mathcal{S}\}$. For later use, we record an observation on the commutant of $\mathcal{G}(A,\Gamma)$.

Proposition 2.4. *The commutant of* $\mathcal{G}(A,\Gamma)$ *in* $\mathcal{B}\big(L^2(\mathbb{R}^2)\big)$ *is abelian.*

Proof. In the notation of [16], $\mathcal{G}(A,\Gamma)$ contains the $\{D_A, T_v \mid v \in N_\Gamma\}$, so $\mathcal{G}(A,\Gamma)'$ is contained in $\{D_A, T_v \mid v \in N_\Gamma\}'$. This latter set is abelian by equation (2) in [16], after implementing the unitary equivalence given by the Fourier transform. □

3. The 3Γ-wavelet representation

With the goal of understanding $\mathcal{G}(A,\Gamma)$ better, we take a closer look at the subgroup of $\mathrm{Aff}(\mathbb{R}^2)$ generated by $[0,A]$ and Γ. For each $\ell \in \mathbb{Z}$, $[0,A]^\ell\Gamma[0,A]^{-\ell}$ is a subgroup of $\mathrm{Aff}(\mathbb{R}^2)$ and

$$\cdots \subseteq [0,A]^2\Gamma[0,A]^{-2} \subseteq [0,A]^1\Gamma[0,A]^{-1} \subseteq \Gamma \subseteq [0,A]^{-1}\Gamma[0,A]^1 \subseteq [0,A]^{-2}\Gamma[0,A]^2 \subseteq \cdots .$$

Let $\Gamma_A = \cup_{\ell\in\mathbb{Z}}[0,A]^\ell\Gamma[0,A]^{-\ell} = \cup_{m=1}^\infty[0,A]^{-m}\Gamma[0,A]^m$. Then Γ_A is a countable subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ such that, if $[x,L] \in \Gamma_A$, then $L \in \mathcal{D}$. Let $\mathcal{N}_A = \mathrm{Trans}(\mathbb{R}^2) \cap \Gamma_A$, the pure translations in Γ_A. There are two subsets of $\mathbb{R}^2$ that are of particular interest to us. Let $T_{\Gamma_A} = \{x \in \mathbb{R}^2 : [x,L] \in \Gamma_A, \text{ for some } L \in \mathcal{D}\}$. Finally, let $N_{\Gamma_A} = \{x \in \mathbb{R}^2 : [x,0] \in \mathcal{N}_A\} = \cup_{k=1}^\infty A^{-k}N_\Gamma$.

Proposition 3.1. *Let Γ be a wallpaper group and let $A \in \mathcal{GL}_2(\mathbb{R})$ be compatible with Γ. Then*

(1) *$\mathcal{N}_A$ is a normal subgroup of Γ_A.*
(2) *$q(\Gamma_A) = \mathcal{D}$.*
(3) *$\Gamma_A/\mathcal{N}_A$ is isomorphic to $\mathcal{D}$.*
(4) *Both N_{Γ_A} and T_{Γ_A} are dense in $\mathbb{R}^2$.*

Proof. (1) and (2) are immediate since $\mathrm{Trans}(\mathbb{R}^2)$ is a normal subgroup of $\mathcal{GL}_2(\mathbb{R})$. Since $\mathcal{N}_A = \{[x,L] \in \Gamma_A : q[x,L] = \mathrm{id}\}$ and q is a homorphism onto $\mathcal{D}$ when restricted to Γ_A, (3) follows. As observed above, $N_{\Gamma_A} = \cup_{k=1}^\infty A^{-k}N_\Gamma$ is dense in $\mathbb{R}^2$. Thus T_{Γ_A} is dense as well. □

In what follows, we will restrict our attention to compatible A in the center of $\mathcal{GL}_2(\mathbb{R})$. This significantly simplifies calculations. For any $d \in \mathbb{N}$, $d \geq 2$, $A = d{\cdot}\mathrm{id}$ is compatible with each of the symmorphic wallpaper groups, because $ALA^{-1} = L$, and $A(ju+kv) = (dj)u+(dk)v \in T_\Gamma^L$ for each $L \in \mathcal{D}$ and all $ju + kv \in T_\Gamma$. However, as we will see in the proof of Proposition 3.2, d must be odd for A to be compatible with a nonsymmorphic group.

Proposition 3.2. *Let $A = d \cdot \mathrm{id}$, with $d \in \mathbb{N}$ odd. Then A is compatible with all 17 of the wallpaper groups.*

Proof. Since for any $[x,L] \in \Gamma$ where Γ is any wallpaper group, we have $ALA^{-1} = L$, we need only verify $Ax \in T_\Gamma^L$ for all $[x,L] \in \Gamma$. This is clearly true if x is of the form $ju + kv$, with $(j,k) \in \mathbb{Z}^2$. The other possibility for $[x,L]$ is $[ju + kv + \frac{1}{2}z, S]$ with $(j,k) \in \mathbb{Z}^2$, where $z = v$ or $z = u + v$, and S is reflection in the vertical axis. In either case, since d is odd,

$Ax = dju + dkv + \frac{d}{2}z$ is of the form $j'u + k'v + \frac{1}{2}z$ with $(j', k') \in \mathbb{Z}^2$, and thus is in T_Γ^S. Note that for d even, Ax is of the form $j'u + k'v$, with $(j', k') \in \mathbb{Z}^2$, which is not in T_Γ^S. □

To further simplify, we will work with the smallest available d; from now on $A = 3 \cdot \mathrm{id}$. There would be no meaningful change in the following if 3 is replaced by any odd integer greater than 1. To acknowledge that $A = 3 \cdot \mathrm{id}$ from now on $\Gamma_A, \mathcal{N}_A, N_{\Gamma_A}$ and T_{Γ_A} are written as $\Gamma_3, \mathcal{N}_3, N_{\Gamma_3}$ and T_{Γ_3}, respectively.

We can use conjugation by $[0, 3 \cdot \mathrm{id}]$ to define an action ϑ of $\mathbb{Z}$ on Γ_3. For $\ell \in \mathbb{Z}$, ϑ_ℓ is defined on Γ_3 by $\vartheta_\ell[x, L] = [0, 3 \cdot \mathrm{id}]^{-\ell}[x, L][0, 3 \cdot \mathrm{id}]^\ell = [3^{-\ell}x, L]$, for all $[x, L] \in \Gamma_3$. We then form the semi-direct product group

$$\Gamma_3 \rtimes_\vartheta \mathbb{Z} = \{\big([x, L], \ell\big) : [x, L] \in \Gamma_3, \ell \in \mathbb{Z}\},$$

equipped with group product

$$\big([x, L], \ell\big)\big([y, M], m\big) = \big([x, L](\vartheta_\ell[y, M]), \ell + m\big) = \big([M^{-1}x + 3^{-\ell}y, LM], \ell + m\big). \tag{2}$$

Note that $\big([x, L], \ell\big)^{-1} = \big([-3^\ell Lx, L^{-1}], -\ell\big)$, for $\big([x, L], \ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$. We will identify Γ_3 with $\{\big([x, L], 0\big) : [x, L] \in \Gamma_3\}$, a normal subgroup of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. Likewise, we identify $\mathcal{N}_3$ with its copy inside $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$.

Proposition 3.3. *$\mathcal{N}_3$ is a normal subgroup of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and $\Gamma_3 \rtimes_\vartheta \mathbb{Z}/\mathcal{N}_3$ is isomorphic to $\mathcal{D} \times \mathbb{Z}$.*

Proof. Considering $\mathcal{N}_3$ as a normal subgroup of Γ_3 for the moment we have, for any $\ell \in \mathbb{Z}$, $\vartheta_\ell[x, \mathrm{id}] = [3^{-\ell}x, \mathrm{id}] \in \mathcal{N}_3$, for any $[x, \mathrm{id}] \in \mathcal{N}_2$. This means $\mathcal{N}_3$ is also normal in the semi-direct product $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. The map Q defined by $Q\big([x, L], \ell\big) = (L, \ell)$ is a homomorophism of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ onto $\mathcal{D} \times \mathbb{Z}$ and $\ker(Q) = \mathcal{N}_3$. This shows that $\Gamma_3 \rtimes_\vartheta \mathbb{Z}/\mathcal{N}_3$ is isomorphic to $\mathcal{D} \times \mathbb{Z}$. □

We will need to factor elements of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ in a particular manner. Although this is just an observation, we state it as a lemma for future reference.

Lemma 3.4. *For $\big([x, L], \ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, we have*

$$\big([x, L], \ell\big) = \big([0, \mathrm{id}], \ell\big)\big([3^\ell x, L], 0\big).$$

Proposition 3.5. *Let Γ be a wallpaper group and let Γ_3, $\mathcal{N}_3$ and T_{Γ_3} be as defined above. The following hold:*

(1) $\mathcal{N}_3 = \left\{\left[\left(\frac{j}{3^\ell}\right)u + \left(\frac{k}{3^\ell}\right)v, \mathrm{id}\right] : (j, k) \in \mathbb{Z}^2, \ell = 0, 1, 2, \cdots\right\}$.

(2) *If* Γ *is symmorphic, then* $T_{\Gamma_3} = \{x \in \mathbb{R}^2 : [x, \mathrm{id}] \in \mathcal{N}_3\}$.
(3) *If* Γ *is nonsymmorphic, then* $T_{\Gamma_3} = \bigcup_{\ell=0}^{\infty} \left(\left\{ \left(\frac{j}{3^\ell}\right) u + \left(\frac{k}{3^\ell}\right) v : (j,k) \in \mathbb{Z}^2 \right\} \cup \left\{ \left(\frac{j}{3^\ell}\right) u + \left(\frac{k}{3^\ell}\right) v + \left(\frac{1}{2}\right) z : (j,k) \in \mathbb{Z}^2 \right\} \right)$, *where* z *is the vector identified in Lemma 2.1.*

Proof. (1) and (2) are clear, so we consider (3). For each integer $\ell \geq 0$, let

$$T_\ell = \{x \in \mathbb{R}^2 : [x, L] \in [0, 3 \cdot \mathrm{id}]^{-\ell} \Gamma [0, 3 \cdot \mathrm{id}]^\ell, \text{ for some } L \in \mathcal{D}\}.$$

We show that

$$T_\ell = \left\{ \left(\frac{j}{3^\ell}\right) u + \left(\frac{k}{3^\ell}\right) v, \left(\frac{j}{3^\ell}\right) u + \left(\frac{k}{3^\ell}\right) v + \left(\frac{1}{2}\right) z : (j,k) \in \mathbb{Z}^2 \right\}, \tag{3}$$

for each $\ell \geq 0$, by induction. When $\ell = 0$, $T_0 = T_\Gamma$ and the claim holds by the choice of z. Suppose the claim holds for some $\ell \geq 0$. For any $x \in T_{\ell+1}$, there exists $L \in \mathcal{D}$ such that $[x, L] \in [0, 3 \cdot \mathrm{id}]^{-\ell-1} \Gamma [0, 3 \cdot \mathrm{id}]^{\ell+1}$. Thus, $[3x, L] = [0, 3 \cdot \mathrm{id}][x, L][0, 3 \cdot \mathrm{id}]^{-1} \in [0, 3 \cdot \mathrm{id}]^{-\ell} \Gamma [0, 3 \cdot \mathrm{id}]^\ell$. By the inductive hypothesis, either $3x = \left(\frac{j}{3^\ell}\right) u + \left(\frac{k}{3^\ell}\right) v$ or $3x = \left(\frac{j}{3^\ell}\right) u + \left(\frac{k}{3^\ell}\right) v + \left(\frac{1}{2}\right) z$, for some $(j,k) \in \mathbb{Z}^2$. Thus, we have that either $x = \left(\frac{j}{3^{\ell+1}}\right) u + \left(\frac{k}{3^{\ell+1}}\right) v$ or $x = \left(\frac{j}{3^{\ell+1}}\right) u + \left(\frac{k}{3^{\ell+1}}\right) v + \left(\frac{1}{6}\right) z$, for some $(j,k) \in \mathbb{Z}^2$. The first alternative is of the correct form. If $x = \left(\frac{j}{3^{\ell+1}}\right) u + \left(\frac{k}{3^{\ell+1}}\right) v + \left(\frac{1}{6}\right) z$, for some $(j,k) \in \mathbb{Z}^2$, we need to consider the two possibilities for z. Either $z = v$ or $z = u + v$. We note that $\frac{1}{6} = \frac{1}{2} - \frac{3^\ell}{3^{\ell+1}}$. So

$$x = \left(\frac{j}{3^{\ell+1}}\right) u + \left(\frac{k - 3^\ell}{3^{\ell+1}}\right) v + \left(\frac{1}{2}\right) z, \text{ if } z = v,$$

and

$$x = \left(\frac{j - 3^\ell}{3^{\ell+1}}\right) u + \left(\frac{k - 3^\ell}{3^{\ell+1}}\right) v + \left(\frac{1}{2}\right) z, \text{ if } z = u + v.$$

Thus, Equation (3) holds for all integers $\ell \geq 0$, and this verifies Condition (3) of the Propostion. $\square$

Although we will not use the following facts, it is interesting to note that $x \to [x, \mathrm{id}]$ embeds T_{Γ_3} as a dense subgroup of $\mathrm{Trans}(\mathbb{R}^2)$ in which $\mathcal{N}_3$ is an index two subgroup. Notice also that $\mathcal{N}_3$ is exactly the intersection of this larger subgroup of $\mathrm{Trans}(\mathbb{R}^2)$ with Γ_3.

In the theory of wavelets with crystal symmetries as "shifts" as developed in [17] the role of the translation unitaries is replaced by $R[x, L]$, with $[x, L] \in \Gamma$. Therefore, the wavelet representation defined in equation (1) of [16] generalizes to the following map of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ into the unitary group of $L^2(\mathbb{R}^2)$.

Definition 3.6. The 3Γ-*wavelet representation* is the map V of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ into the group of unitary operators on $L^2(\mathbb{R}^2)$ defined by, for $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$,

$$V\big([x,L],\ell\big) = R[x,L]D_3^\ell,$$

where $D_3 g(y) = 3g(3y)$, for $y \in \mathbb{R}^2$ and $g \in L^2(\mathbb{R}^2)$.

Proposition 3.7. *Let Γ be a wallpaper group and let $A = 3 \cdot \mathrm{id}$. Then the 3Γ-wavelet representation is a faithful unitary representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ on $L^2(\mathbb{R}^2)$. Moreover,*

$$V\big(\Gamma_3 \rtimes_\vartheta \mathbb{Z}\big) = \mathcal{G}(A,\Gamma).$$

Proof. Direct computation shows that, for any $[y,M] \in \Gamma_3$,

$$D_3 R[y,M] = R[3^{-1}y, M]D_3. \tag{4}$$

Using this repeatedly, we have, for $\big([x,L],\ell\big), \big([y,M],m\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$,

$$V\big([x,L],\ell\big)V\big([y,M],m\big) = R[x,L]D_3^\ell R[y,M]D_3^m = R[x,L]R[3^{-\ell}y,M]D_3^{\ell+m}.$$

But R is a homomorphism, so $R[x,L]R[3^{-\ell}y,M] = R\big([x,L][3^{-\ell}y,M]\big) = R[M^{-1}x+3^{-\ell}y, LM]$ and, thus, $V\big([x,L],\ell\big)V\big([y,M],m\big) = R[M^{-1}x + 3^{-\ell}y, LM]D_3^{\ell+m}$. Using (2), we see that V is a homomorphism of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ into the unitary group of $L^2(\mathbb{R}^2)$. It is clear that the image of V is the smallest group of unitary operators on $L^2(\mathbb{R}^2)$ containing $R[0,A] = D_3^{-1}$ and the set $R(\Gamma)$. That is, the image of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ under V is $\mathcal{G}(A,\Gamma)$. Note that, for $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, if $R[x,L]D_3^\ell$ is the identity operator, then $R[x,L] = D_3^{-\ell}$, from which one can verify that $x = 0$, $L = \mathrm{id}$, and $\ell = 0$. This implies V is faithful. ☐

It will be useful to convert V to an equivalent representation $\widehat{V}$ using the Fourier transform. We use the following form of the Fourier transform. For $g \in L^1(\mathbb{R}^2)$,

$$\mathcal{F}(g)(\omega) = \widehat{g}(\omega) = \int_{\mathbb{R}^2} g(x)e^{-2\pi i\langle x,\omega\rangle}dx, \text{ for all } \omega \in \mathbb{R}^2.$$

For $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, let $\widehat{V}\big([x,L],\ell\big) = \mathcal{F}V\big([x,L],\ell\big)\mathcal{F}^{-1}$. A direct computation provides an explicit formula for $\widehat{V}$.

Proposition 3.8. *For any $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and any $h \in L^2(\mathbb{R}^2)$,*

$$\widehat{V}\big([x,L],\ell\big)h(\omega) = 3^{-\ell}e^{-2\pi i\langle x, L^{-1}\omega\rangle}h(3^{-\ell}L^{-1}\omega), \text{ for all } \omega \in \mathbb{R}^2.$$

Our primary goal is to decompose $\widehat{V}$ as a direct integral of irreducible representations that we describe in the next section. Note that Proposition 3.7 combined with Proposition 2.4 implies that the representation $\widehat{V}$ is multiplicity-free. Thus the direct integral decomposition into irreducibles constructed in the next section will be essentially unique ([19], Theorem, p. 117).

4. A family of irreducible representations of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$

The components in our decomposition of the 3Γ-wavelet representation will be certain irreducible representations of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ each of which is induced from a character of $\mathcal{N}_3$.

The normal subgroup $\mathcal{N}_3$ is a countable discrete abelian group. Its dual, $\widehat{\mathcal{N}_3}$, is a compact abelian group. There is a distinguished subset of $\widehat{\mathcal{N}_3}$ consisting of restrictions of continuous characters of $\mathbb{R}^2$ to N_3, then moved to $\mathcal{N}_3$. For each $\omega \in \mathbb{R}^2$, define $\chi_\omega : \mathcal{N}_3 \to \mathbb{T}$ by

$$\chi_\omega\big([x,\mathrm{id}],0\big) = e^{-2\pi i \langle x,\omega\rangle}, \quad \text{for all } \big([x,\mathrm{id}],0\big) \in \mathcal{N}_3.$$

Because N_3 is dense in $\mathbb{R}^2$, $\chi_\omega = \chi_{\omega'}$ if and only if $\omega = \omega'$, for $\omega, \omega' \in \mathbb{R}^2$.

Proposition 4.1. *The map $\omega \to \chi_\omega$ is a continuous one-to-one homomorphism of $\mathbb{R}^2$ onto a dense subgroup of $\widehat{\mathcal{N}_3}$.*

Proof. The facts that $\omega \to \chi_\omega$ is one-to-one and a homomorphism are obvious. Since $\mathcal{N}_3$ is being considered with the discrete topology, the topology of $\widehat{\mathcal{N}_3}$ is the topology of pointwise convergence, so continuity is easy. Let $\Omega = \{\chi_\omega : \omega \in \mathbb{R}^2\}$, a subgroup of $\widehat{\mathcal{N}_3}$. For any $[x,\mathrm{id}] \in \mathcal{N}_3$, if $[x,\mathrm{id}] \neq [0,\mathrm{id}]$, then there exists $\omega \in \mathbb{R}^2$ such that $\chi_\omega[x,\mathrm{id}] \neq 1$. Thus, the annihilator of Ω in $\mathcal{N}_3$ is $\{[0,\mathrm{id}]\}$, which means that the double annihilator is $\widehat{\mathcal{N}_3}$. But the double annihilator is $\overline{\Omega}$, see [11]. This shows that Ω is dense in $\widehat{\mathcal{N}_3}$. □

Thus, we can think of Ω as a copy of $\mathbb{R}^2$ equipped with a weaker topology sitting densely in $\widehat{\mathcal{N}_3}$.

We will now induce the characters in Ω to $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. There are several different versions of induced representations that can all be shown to be unitarily equivalent to one another. Here we use the version given in [15] Chapter 2 and in [9] Chapter 6.1, Remark 2, p. 155. The basis for these descriptions appeared in [18].

This general definition for induced representations applies to a representation π, acting in the Hilbert space $\mathcal{H}_\pi$, of a closed subgroup H of a locally compact group G. In general, the definition involves the Radon-Nikodym derivative λ for a quasi-invariant measure under the left action of G on G/H. The induced representation acts in a Hilbert space of square integrable functions $f : G \mapsto \mathcal{H}_\pi$ satisfying $f(xh) = \pi(h^{-1})(f(x))\ \forall h \in H$, by

$$U^\pi(x)(f)(y) = \sqrt{\lambda(x^{-1}, yH)}f(x^{-1}y),$$

for $x, y \in G$. We are interested in the following special case:

Definition 4.2. For each $\chi_\omega \in \Omega$, let $U^\omega = \operatorname{Ind}_{\mathcal{N}_A}^{\Gamma_A \rtimes_\vartheta \mathbb{Z}} \chi_\omega$, the representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ induced from the representation χ_ω of $\mathcal{N}_3$.

Since $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is discrete $\mathcal{N}_3$ is an open subgroup. Also, the χ_ω are one dimensional representations of $\mathcal{N}_3$. This makes inducing easier, with details for inducing from an open subgroup worked out in [15], Section 2.1. We will provide an explicit formula for U^ω below. However, both to develop the form we need, and to determine which U^ω are irreducible and when two of them are equivalent, we first need to understand the action of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}/\mathcal{N}_3 = \mathcal{D} \times \mathbb{Z}$ on $\widehat{\mathcal{N}_3}$.

Elements of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ act on the normal subgroup $\mathcal{N}_3$ by conjugation. That is, for $\big([x, L], \ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and $\big([y, \mathrm{id}], 0\big) \in \mathcal{N}_3$,

$$\begin{aligned}\big([x, L], \ell\big) \cdot \big([y, \mathrm{id}], 0\big) &= \big([x, L], \ell\big)\big([y, \mathrm{id}], 0\big)\big([x, L], \ell\big)^{-1} \\ &= \big([x + 3^{-\ell}y, L], \ell\big)\big([-3^\ell Lx, L^{-1}], -\ell\big) = \big([3^{-\ell}Ly, \mathrm{id}], 0\big).\end{aligned}$$

As expected, this action only depends on the coset of $\mathcal{N}_3$ containing $\big([x, L], \ell\big)$. Thus, it is actually an action of $\mathcal{D} \times \mathbb{Z}$. We write $(L, \ell) \cdot [y, \mathrm{id}] = [3^{-\ell}Ly, \mathrm{id}]$, for each $(L, \ell) \in \mathcal{D} \times \mathbb{Z}$ and $\big([y, \mathrm{id}], 0\big) \in \mathcal{N}_3$. This then determines an action of $\mathcal{D} \times \mathbb{Z}$ on $\widehat{\mathcal{N}_3}$. For $(L, \ell) \in \mathcal{D} \times \mathbb{Z}$ and $\chi \in \widehat{\mathcal{N}_3}$, define $(L, \ell) \cdot \chi \in \widehat{\mathcal{N}_3}$ by

$$((L, \ell) \cdot \chi)\big([y, \mathrm{id}], 0\big) = \chi\big((L^{-1}, -\ell) \cdot ([y, \mathrm{id}], 0)\big) = \chi\big([3^\ell L^{-1}y, \mathrm{id}], 0\big), \text{ for all, } \big([y, \mathrm{id}], 0\big) \in \mathcal{N}_3.$$

If $\omega \in \mathbb{R}^2$, then $(L, \ell) \cdot \chi_\omega = \chi_{(3^\ell L^{-1})^t \omega}$. Note that $(3^\ell L^{-1})^t = 3^\ell L$, since $\mathcal{D} \subseteq \mathcal{O}_2$. Thus, $(L, \ell) \cdot \chi_\omega = \chi_{3^\ell L\omega}$, for $(L, \ell) \in \mathcal{D} \times \mathbb{Z}$, and Ω is invariant under the action of $\mathcal{D} \times \mathbb{Z}$ on $\widehat{\mathcal{N}_3}$. To understand the orbit structure in Ω under the action of $\mathcal{D} \times \mathbb{Z}$, it suffices to describe the orbit structure in $\mathbb{R}^2$ under the action of $\mathcal{D} \times \mathbb{Z}$. For each $\omega \in \mathbb{R}^2$, let $\mathcal{D}(\omega) = \{L\omega : L \in \mathcal{D}\}$, $(\mathcal{D} \times \mathbb{Z})(\omega) = \{3^\ell L\omega : (L, \ell) \in \mathcal{D} \times \mathbb{Z}\}$ and $\mathcal{D}_\omega = \{L \in \mathcal{D} : L\omega = \omega\}$. We call $\mathcal{D}(\omega)$ a $\mathcal{D}$-*orbit*, $(\mathcal{D} \times \mathbb{Z})(\omega)$ a $(\mathcal{D} \times \mathbb{Z})$-*orbit* and $\mathcal{D}_\omega$ the *stability subgroup* of ω in $\mathcal{D}$. Note that the stability subgroup of ω in $\mathcal{D} \times \mathbb{Z}$ is $\{(L, 0) : L \in \mathcal{D}_\omega\}$. The following proposition is a result of Theorem 2.6 and Proposition 2.8 of [15].

Proposition 4.3. *Let $\omega, \omega' \in \mathbb{R}^2$. Then U^ω is irreducible if and only if $\mathcal{D}_\omega = \{\mathrm{id}\}$ and $U^{\omega'}$ is equivalent to U^ω if and only if $(\mathcal{D} \times \mathbb{Z})(\omega') = (\mathcal{D} \times \mathbb{Z})(\omega)$.*

For any $X \subseteq \mathbb{R}^2$ and $(L, \ell) \in \mathcal{D} \times \mathbb{Z}$, let $3^\ell LX = \{3^\ell L\omega : \omega \in X\}$.

Definition 4.4. A subset X of $\mathbb{R}^2$ is called a *weak $3\mathcal{D}$-cross-section* if

(a) X is Borel.

(b) For $(L,\ell),(M,m) \in \mathcal{D}\times\mathbb{Z}$, $(L,\ell)\neq(M,m)$ implies $\left(3^\ell LX\right)\cap\left(3^m MX\right)=\emptyset$.

(c) $m(\mathbb{R}^2\setminus\cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}}\,3^\ell LX) \;=\; 0$, where m denotes Lebesgue measure on $\mathbb{R}^2$.

Note that if X is a weak $3\mathcal{D}$-cross-section, then $\cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}}3^\ell LX$ will be dense in $\mathbb{R}^2$.

Proposition 4.5. *Let $\mathcal{D}$ be the point group of a wallpaper group Γ. Then a weak $3\mathcal{D}$-cross-section exists.*

Proof. If $\mathcal{D}=\mathcal{C}_k$, for some $k\in\{1,2,3,4,6\}$, then $\mathcal{D}_\omega$ is trivial, for all $\omega\neq 0$. Let $Y=\{(r,0): r\in\mathbb{R}, r>0\}$ and $Z=\cup\{\rho(\theta)\omega:\omega\in Y, 0<\theta<\frac{2\pi}{k}\}$. Then, for each $\omega\in\mathbb{R}^2$, $\omega\neq 0$, $\mathcal{D}(\omega)\cap(Y\cup Z)$ is a singleton. On the other hand, if $\mathcal{D}$ contains a reflection, then, with Y as just defined, there exist $0\leq\theta_1<\theta_2<2\pi$ such that $\{\rho(\theta)\omega:\omega\in Y,\theta_1\leq\theta\leq\theta_2\}$ contains exactly one member from each nonzero $\mathcal{D}$-orbit in $\mathbb{R}^2$. In particular, if $\mathcal{D}=\mathcal{D}_k$ contains a reflection in the line determined by $(\cos\theta,\sin\theta)$, as well as rotations by multiples of $\frac{2\pi}{k}$, then θ_1 can be taken to be θ, and θ_2 to be $\theta+\frac{\pi}{k}$. Moreover, if $Z=\{\rho(\theta)\omega:\omega\in Y,\theta_1<\theta<\theta_2\}$, then $\mathcal{D}_\omega$ is trivial for each $\omega\in Z$. In either case, Z is open, $\cup L\in\mathcal{D}LZ$ is dense in $\mathbb{R}^2$ and $LZ\cap MZ=\emptyset$, if $L\neq M$. Moreover, $\omega\in Z$ implies $r\omega\in Z$, for all $r>0$. If $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$, let

$$X=\{\omega\in Z: 1\leq\|\omega\|<3\}.$$

Then X is Borel, $3^\ell X\cap 3^m X=\emptyset$, for $\ell\neq m$, and one easily checks that $m(\mathbb{R}^2\setminus\cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}}\, 3^\ell LX) \;=\; 0$. Therefore, X is a weak $3\mathcal{D}$-cross-section. $\square$

One interesting source of weak $3\mathcal{D}$-cross-sections are 3Γ-wavelet sets, that is, Borel sets $W\subseteq\mathbb{R}^2$ such that the characteristic function $\mathbf{1}_W$ is the Fourier transform of an $A\Gamma$-wavelet, where $A=3\cdot\mathrm{id}$. To be the Fourier transform of an $A\Gamma$-wavelet, this characteristic function must be orthogonal to its dilates by nontrivial powers of A as well as to its transformations by elements of $\mathcal{F}\Gamma\mathcal{F}^{-1}$. The characteristic function must also have the property that dilates of these transformations form an orthonormal basis for $L^2(\mathbb{R}^2)$. These properties are easily seen to imply conditions (b) and (c) of Definition 4.4. This is discussed further in [23], where $A\Gamma$-wavelet sets are shown to exist for all wallpaper groups and all integer dilations.

Let us briefly recall the concept of weak equivalence for sets of unitary representations. Let G be a locally compact group and let $C^*(G)$ denote the group C^*-algebra of G. See Section 7.1 of [9] for basic information on $C^*(G)$. Any unitary representation π of G determines a unique nondegenerate $*$-representation, also denoted π, of $C^*(G)$. This correspondence of representations preserves irreducibility and equivalence. For a unitary representation π of

G, $\mathrm{Ker}(\pi)$ denotes the kernel of π as a nondegenerate $*$-representation of $C^*(G)$, a closed $*$-ideal in $C^*(G)$.

Definition 4.6. Let $\mathcal{S}$ and $\mathcal{T}$ be sets of unitary representations of a locally compact group G. We say that $\mathcal{S}$ and $\mathcal{T}$ are *weakly equivalent* if

$$\cap\{\mathrm{Ker}(\sigma) : \sigma \in \mathcal{S}\} = \cap\{\mathrm{Ker}(\tau) : \tau \in \mathcal{T}\}.$$

If π is a single unitary representation of G and $\mathcal{S}$ is weakly equivalent to $\{\pi\}$, then we simply say $\mathcal{S}$ is weakly equivalent to π.

Theorem 4.7. *Let Γ be a wallpaper group with point group $\mathcal{D}$. If X is a weak $3\mathcal{D}$-cross-section, then $\{U^\omega : \omega \in X\}$ is weakly equivalent to the left regular representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$.*

Proof. Recall from Proposition 4.1 that $\Omega = \{\chi_\omega : \omega \in \mathbb{R}^2\}$ is dense in $\widehat{\mathcal{N}_3}$. Thus, the set $\Omega_X = \{\chi_\omega : \omega \in \cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}} 3^\ell LX\}$ is dense in $\widehat{\mathcal{N}_3}$.

This implies that the left regular representation of $\mathcal{N}_3$ is weakly equivalent to Ω_X. Now apply Corollary 5.41 of [15] to conclude that the left regular representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is weakly equivalent to $\{U^\omega : \omega \in \cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}} 3^\ell LX\}$. But, for every $\omega \in \cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}} 3^\ell LX$, there exists an $\omega' \in X$ so that $U^{\omega'} \sim U^\omega$. This means that the left regular representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is weakly equivalent to $\{U^\omega : \omega \in X\}$. $\square$

Remark 4.8. *Since $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is amenable (see, for example, [26]), the left regular representation is a faithful representation of $C^*(\Gamma_3 \rtimes_\vartheta \mathbb{Z})$ by Hulanicki's Theorem [12]. Therefore, if X is a weak $3\mathcal{D}$-cross-section, then $\{U^\omega : \omega \in X\}$ is a faithful family of irreducible representations of $C^*(\Gamma_3 \rtimes_\vartheta \mathbb{Z})$.*

In order to get an explicit expression for the U^ω, we will need to fix a *section* from the quotient group $\mathcal{D} \times \mathbb{Z}$ into $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. That is, we fix a map $\gamma : \mathcal{D} \times \mathbb{Z} \to \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, satisfying $(Q \circ \gamma)(L, \ell) = (L, \ell)$ as follows:

Definition 4.9. For $(L, \ell) \in \mathcal{D} \times \mathbb{Z}$, let

$$\gamma(L, l) = \begin{cases} ([0, L], l) & L \in \mathcal{D}^0 \\ ([\frac{3^{-l}}{2} z, L], l) & L \notin \mathcal{D}^0 \end{cases},$$

where $z = v$ if Γ is either pg or $pmg2$ and $z = u + v$ if Γ is either $pgg2$ or $p4mg$.

This is legitimate since $[\frac{3^{-l}}{2} z, L] = [0, 3 \cdot \mathrm{id}]^{-l} [\frac{1}{2} z, L][0, 3 \cdot \mathrm{id}]^l \in \Gamma_3$.

Each member of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ can be uniquely written in the form $\gamma(L,\ell)\big([x,\mathrm{id}],0\big)$ with $(L,\ell) \in \mathcal{D} \times \mathbb{Z}$ and $\big([x,\mathrm{id}],0\big) \in \mathcal{N}_3$. Indeed, for $\big([y,M],m\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$,

$$\big([y,M],m\big)] = \gamma(M,m)\big(\gamma(M,m)^{-1}\big([y,M],m\big)\big) \quad \text{and} \quad \gamma(M,m)^{-1}\big([y,M],m\big) \in \mathcal{N}_3.$$

Since $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is discrete, inducing a one dimensional representation, such as χ_ω, for $\omega \in \mathbb{R}^2$ from $\mathcal{N}_3$ up to $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ takes a relatively simple form (see Section 2.1 of [15]). Let

$$\mathcal{H}_{U^\omega} = \{\xi : \Gamma_3 \rtimes_\vartheta \mathbb{Z} \to \mathbb{C} \text{ satisfying (a) and (b)}\}, \text{ where}$$

(a) $\xi\big(\gamma(L,\ell)\big([x,\mathrm{id}],0\big)\big) = \chi_\omega\big([-x,\mathrm{id}],0\big)\xi\big(\gamma(L,\ell)\big)$, for all $(L,\ell) \in \mathcal{D}\times\mathbb{Z}$, $\big([x,\mathrm{id}],0\big) \in \mathcal{N}_3$.
(b) $\sum_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}} |\xi\big(\gamma(L,\ell)\big)|^2 < \infty$.

We equip $\mathcal{H}_{U^\omega}$ with the inner product given by $\langle \xi,\eta\rangle = \sum_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}} \xi\big(\gamma(L,\ell)\big)\overline{\eta\big(\gamma(L,\ell)\big)}$, for $\xi,\eta \in \mathcal{H}_{U^\omega}$. The induced representation, U^ω is realized on $\mathcal{H}_{U^\omega}$. For $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, $U^\omega\big([x,L],\ell\big)$ is the unitary operator on $\mathcal{H}_{U^\omega}$ defined by

$$U^\omega\big([x,L],\ell\big)\xi\big([y,M],m\big) = \xi\Big(\big([x,L],\ell\big)^{-1}\big([y,M],m\big)\Big),$$

for all $\big([y,M],m\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, $\xi \in \mathcal{H}_{U^\omega}$. It is often useful to work with a representation that is unitarily equivalent to U^ω obtained by noticing that $W : \ell^2(\mathcal{D}\times\mathbb{Z}) \to \mathcal{H}_{U^\omega}$ given by $Wf\big(\gamma(L,\ell)\big([x,\mathrm{id}],0\big)\big) = \chi_\omega(-x)f(L,\ell)$, for $\gamma(L,\ell)\big([x,\mathrm{id}],0\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, $f \in \ell^2(\mathcal{D}\times\mathbb{Z})$, is a unitary map of $\ell^2(\mathcal{D}\times\mathbb{Z})$ onto $\mathcal{H}_{U^\omega}$. Define

$$\sigma_\omega\big([x,L],\ell\big) = W^{-1}U^\omega\big([x,L],\ell\big)W, \quad \text{for all } \big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}.$$

Although σ_ω depends on the section γ, we suppress its role in the notation. We note that $W^{-1}\xi(L,\ell) = \xi\big(\gamma(L,\ell)\big)$, for $(L,\ell) \in \mathcal{D}\times\mathbb{Z}$, $\xi \in \mathcal{H}_{U^\omega}$.

Proposition 4.10. *For $\omega \in \mathbb{R}^2$, σ_ω is given by*

$$\sigma_\omega\big([x,L],\ell\big)f(M,m) = \chi_\omega\big(\gamma(M,m)^{-1}\big([x,L],\ell\big)\gamma(L^{-1}M,m-\ell)\big)f(L^{-1}M,m-\ell),$$

for $(M,m) \in \mathcal{D}\times\mathbb{Z}$, $f \in \ell^2(\mathcal{D}\times\mathbb{Z})$, and $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$.

Proof. Fix $(M,m) \in \mathcal{D}\times\mathbb{Z}$ and $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$. Then

$$\big([x,L],\ell\big)^{-1}\gamma(M,m) = \gamma(L^{-1}M,m-\ell)\Big(\gamma(L^{-1}M,m-\ell)^{-1}\big([x,L],\ell\big)^{-1}\gamma(M,m)\Big).$$

Observe that $Q\Big(\gamma(L^{-1}M,m-\ell)^{-1}\big([x,L],\ell\big)^{-1}\gamma(M,m)\Big) = (L^{-1}M,m-\ell)^{-1}(L^{-1},-\ell)(M,m) = (\mathrm{id},0)$, since Q is a homomorphism of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ onto $\mathcal{D}\times\mathbb{Z}$. Thus

$$\gamma(L^{-1}M,m-\ell)^{-1}\big([x,L],\ell\big)^{-1}\gamma(M,m) \in \mathcal{N}_3.$$

Therefore, for any $f \in \ell^2(\mathcal{D} \times \mathbb{Z})$,

$$\begin{aligned}
\sigma_\omega\big([x,L],\ell\big)f(M,m) &= W^{-1}U^\omega\big([x,L],\ell\big)Wf(M,m) = U^\omega\big([x,L],\ell\big)Wf\big(\gamma(M,m)\big) \\
&= Wf(\big([x,L],\ell\big)^{-1}\gamma(M,m)) \\
&= Wf\left(\gamma(L^{-1}M,m-\ell)\big(\gamma(L^{-1}M,m-\ell)^{-1}\big([x,L],\ell\big)^{-1}\gamma(M,m)\big)\right) \\
&= \chi_\omega\big(\gamma(M,m)^{-1}\big([x,L],\ell\big)\gamma(L^{-1}M,m-\ell)\big)Wf\left(\gamma(L^{-1}M,m-\ell)\right) \\
&= \chi_\omega\big(\gamma(M,m)^{-1}\big([x,L],\ell\big)\gamma(L^{-1}M,m-\ell)\big)f(L^{-1}M,m-\ell),
\end{aligned}$$

as asserted. $\square$

Using Definition 4.9, and the fact that $\chi_\omega\big([y,\mathrm{id}],0\big) = e^{-2\pi i\langle y,\omega\rangle}$ for $\big([y,\mathrm{id}],0\big) \in \mathcal{N}_3$, the formula for σ_ω simplifies as follows:

Corollary 4.11. *If Γ is a wallpaper group and $\omega \in \mathbb{R}^2$, then*

$$\sigma_\omega\big([x,L],\ell\big)f(M,m) = \begin{cases} e^{-2\pi i\langle x,3^mL^{-1}M\omega\rangle}f((L^{-1}M,m-\ell)) & L \in \mathcal{D}^0 \\ e^{-\pi i\langle z,\omega\rangle}e^{-2\pi i\langle x,3^mL^{-1}M\omega\rangle}f((L^{-1}M,m-\ell)) & L \notin \mathcal{D}^0, M \in \mathcal{D}^0 \\ e^{\pi i\langle z,\omega\rangle}e^{-2\pi i\langle x,3^mL^{-1}M\omega\rangle-\frac{1}{2}z}f((L^{-1}M,m-\ell)) & L \notin \mathcal{D}^0, M \notin \mathcal{D}^0 \end{cases},$$

for $(M,m) \in \mathcal{D} \times \mathbb{Z}$, $f \in \ell^2(\mathcal{D} \times \mathbb{Z})$, and $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$.

Now, suppose that X is a weak $3\mathcal{D}$-cross-section. Then $\cup_{(L,\ell)\in\mathcal{D}\times\mathbb{Z}}3^\ell LX$ is dense in $\mathbb{R}^2$. Moreover, distinct points in X lie in distinct $\mathcal{D} \times \mathbb{Z}$ orbits and the orbit of each point in X is free. In light of Proposition 4.3, $\{\sigma_\omega : \omega \in X\}$ consists of inequivalent irreducible representations of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and Theorem 4.7 says there are enough of them to be weakly equivalent to the left regular representation. We now form the direct integral of the σ_ω with respect to Lebesgue measure of $\mathbb{R}^2$ restricted to X. For a comprehensive treatment of the the theory of direct integrals see [25] or Chapter 14 of [13]. Also, Section 7.4 of [9] provides details for direct integrals of unitary representations.

Each σ_ω, for $\omega \in X$, acts on the same Hilbert space, $\ell^2(\mathcal{D} \times \mathbb{Z})$. This makes it relatively easy to describe the Hilbert space of their direct integral with respect to Lebesgue measure restricted to X. First, we note that a function $F : X \to \ell^2(\mathcal{D} \times \mathbb{Z})$ is called *measurable*, in this context, if $\omega \to \langle F(\omega),\eta\rangle$ is Borel measurable on X, for each $\eta \in \ell^2(\mathcal{D} \times \mathbb{Z})$. Then

$$L^2\big(X,\ell^2(\mathcal{D} \times \mathbb{Z})\big) = \left\{F : X \to \ell^2(\mathcal{D} \times \mathbb{Z}) \,\middle|\, F \text{ is measurable and } \int_X \|F(\omega)\|^2 d\omega < \infty\right\}$$

is the Hilbert space we need. By definition, the direct integral representation, denoted $\int_X^\oplus \sigma_\omega \, d\omega$, acts as follows: For $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and $F \in L^2\big(X, \ell^2(\mathcal{D}\times\mathbb{Z})\big)$,

$$\left[\left(\int_X^\oplus \sigma_\omega \, d\omega\right)\big([x,L],\ell\big)F\right](\omega') = \sigma_{\omega'}\big([x,L],\ell\big)F(\omega'), \quad \text{for a.e. } \omega' \in X. \tag{5}$$

5. A decomposition of the wavelet representation

To decompose the wavelet representation in terms of the family of irreducibles developed in the previous section, we need to conjugate $\widehat{V}$ by a map that breaks $\mathbb{R}^2$ up into a product $X \times \mathcal{D} \times \mathbb{Z}$, where X is a weak $3\mathcal{D}-$cross-section. Because of the extra factor in the induced representations for group elements involving a glide, this conjugating map will include the twist $c : X \times \mathcal{D} \mapsto \mathbb{C}$, defined by

$$c(\omega, L) = \begin{cases} e^{-\frac{\pi i \langle z,\omega\rangle}{2}} & \text{if } L \in \mathcal{D}^0 \\ e^{\frac{\pi i \langle z,\omega\rangle}{2}} & \text{if } L \notin \mathcal{D}^0 \end{cases}, \tag{6}$$

where z is the vector identified in Lemma 2.1.

Given a weak $3\mathcal{D}-$cross-section X, define the map $\rho : L^2(\mathbb{R}^2) \mapsto L^2(X \times \mathcal{D} \times \mathbb{Z}, m \times \nu)$, where m is Lebesgue measure restricted to X, and ν is counting measure on the countable discrete space $\mathcal{D} \times \mathbb{Z}$, by

$$[\rho(\phi)](\omega, M, j) = 3^j c(\omega, M)\phi(3^j M(\omega)),$$

for $\omega \in X$, $j \in \mathbb{Z}$, and $M \in \mathcal{D}$. Then ρ is a Hilbert space isomorphism whose inverse is given by

$$[\rho^{-1}(f)](\xi) = \sum_k 3^{-k} \sum_{M' \in \mathcal{D}} c(-3^{-k}M'^{-1}\xi, M')\mathbf{1}_{(M'(X))}(3^{-k}\xi)[f(3^{-k}M'^{-1}\xi, M', k)], \tag{7}$$

for $\xi \in \mathbb{R}^2$ and $f \in L^2(X \times \mathcal{D} \times \mathbb{Z})$. If $\widetilde{V}$ is defined by $\widetilde{V}\big([x,L],\ell\big) = \rho \widehat{V}\big([x,L],\ell\big)\rho^{-1}$, for all $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$, we get a representation whose detailed action is given in the following proposition.

Proposition 5.1. *For $\big([x,L],\ell\big) \in \Gamma_3 \rtimes_\vartheta \mathbb{Z}$ and $f \in L^2(X \times \mathcal{D} \times \mathbb{Z})$,*

$$\begin{aligned} &[\widetilde{V}\big([x,L],\ell\big)(f)](\omega, M, j) \\ &\quad = \begin{cases} e^{-2\pi i \langle x, 3^j L^{-1}M(\omega)\rangle} f(\omega, L^{-1}M, j-\ell) & L \in \mathcal{D}^0 \\ e^{-\pi i \langle z,\omega\rangle} e^{-2\pi i \langle x, 3^j L^{-1}M(\omega)\rangle} f(\omega, L^{-1}M, j-\ell)] & L \notin \mathcal{D}^0,\ M \in \mathcal{D}^0 \\ e^{\pi i \langle z,\omega\rangle} e^{-2\pi i \langle x, 3^j L^{-1}M(\omega)\rangle} f(\omega, L^{-1}M, j-\ell) & L \notin \mathcal{D}^0,\ M \notin \mathcal{D}^0 \end{cases}, \end{aligned}$$

for a.e. $(\omega, M, j) \in X \times \mathcal{D} \times \mathbb{Z}$.

Proof. We simply compute:

$$
\begin{aligned}
&[\widetilde{V}\big([x,L],\ell\big)(f)](\omega,M,j)\\
&\quad = 3^j c(\omega,M)[\widehat{V}([x,L],\ell)(\rho^{-1}(f))](3^j M(\omega))\\
&\quad = 3^{(j-\ell)} c(\omega,M) e^{-2\pi i\langle x, 3^j L^{-1}M(\omega)\rangle}[\rho^{-1}(f)](3^{j-\ell}L^{-1}M(\omega))\\
&\quad = 3^{(j-\ell)} c(\omega,M) e^{-2\pi i\langle x, 3^j L^{-1}M(\omega)\rangle} 3^{-(j-\ell)} c(-\omega, L^{-1}M)) f(\omega, L^{-1}M, j-\ell)\\
&\quad = \begin{cases} e^{-2\pi i\langle x, 3^j L^{-1}M(\omega)\rangle} f(\omega, L^{-1}M, j-\ell) & L\in\mathcal{D}^0\\ e^{-\pi i\langle z,\omega\rangle} e^{-2\pi i\langle x, 3^j L^{-1}M(\omega)\rangle} f(\omega, L^{-1}M, j-\ell) & L\notin\mathcal{D}^0,\ M\in\mathcal{D}^0\\ e^{\pi i\langle z,\omega\rangle} e^{-2\pi i\langle x, 3^j L^{-1}M(\omega)\rangle} f(\omega, L^{-1}M, j-\ell) & L\notin\mathcal{D}^0,\ M\notin\mathcal{D}^0 \end{cases}.
\end{aligned}
$$

Here the second to last step follows from Equation (7) since the only nonzero summand of $\rho^{-1}(f)(3^{j-l}L^{-1}M(\omega))$ there is for $k = j-\ell$ and $M' = L^{-1}M$. The last step fills in the definition of c from Equation(6). $\square$

Now we are ready for the main theorem of the paper.

Theorem 5.2. *Let Γ be a wallpaper group. The 3Γ-wavelet representation V of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is equivalent to a direct integral of irreducible representations induced from characters of the normal abelian subgroup $\mathcal{N}_3$.*

Proof. Since $\widetilde{V}$ is equivalent to V, it suffices to show that the representation $\widetilde{V}$ is equivalent to $\int_X^\oplus \sigma_\omega\, d\omega$. We note that the map $W : L^2\big(X, \ell^2(\mathcal{D}\times\mathbb{Z})\big) \to L^2(X\times\mathcal{D}\times\mathbb{Z})$ given by $WF(\omega,L,\ell) = \big(F(\omega)\big)(L,\ell)$, for all $(\omega,L,\ell)\in X\times\mathcal{D}\times\mathbb{Z}$ and $F\in L^2\big(X,\ell^2(\mathcal{D}\times\mathbb{Z})\big)$, is a Hilbert space isomorphism, which is easily checked. Using (5) and the explicit formulas in Corollary 4.11 and Proposition 5.1, one verifies directly that

$$W\left[\left(\int_X^\oplus \sigma_\omega d\omega\right)\big([x,L],\ell\big)\right]W^{-1} = \widetilde{V}\big([x,L],\ell\big),$$

for all $\big([x,L],\ell\big)\in\Gamma_3\rtimes_\vartheta\mathbb{Z}$. This completes the proof. $\square$

We refer to Chapter 6 of [13] for the definition of Type I von Neumann algebras. If G is a locally compact group and π is a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$, then the von Neumann algebra generated by π is $\pi(G)''$, the double commutant of $\pi(G)$ inside $\mathcal{B}(\mathcal{H}_\pi)$. If $\pi(G)''$ is a Type I von Neumann algebra, then π is called a *Type I representation.*

Proposition 5.3. *Let Γ be a wallpaper group. Then the 3Γ-wavelet representation V of $\Gamma_3\rtimes_\vartheta\mathbb{Z}$ is a Type I representation.*

Proof. By Proposition 3.7, $V\big(\Gamma_3 \rtimes_\vartheta \mathbb{Z}\big) = \mathcal{G}(A,\Gamma)$, where $A = 3 \cdot \mathrm{id}$. By Proposition 2.4, $V\big(\Gamma_3 \rtimes_\vartheta \mathbb{Z}\big)'$ is abelian and, thus, a Type I von Neumann algebra. By Theorem 9.1.3 of [13], its commutant, $V\big(\Gamma_3 \rtimes_\vartheta \mathbb{Z}\big)''$, is also Type I. That is, V is a Type I representation. □

The 3Γ-wavelet representation V is, in some sense, a natural representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. Another natural representation is the left regular representation. Our final proposition concerns the relationship between these two representations.

Proposition 5.4. *Let Γ be a wallpaper group. Then the 3Γ-wavelet representation V and the left regular representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ are weakly equivalent but not equivalent.*

Proof. The proof of Theorem 3.1 in [16] adapts to this situation to show that V is weakly equivalent to the left regular representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$. We note that $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ has no abelian subgroup of finite index, as is easily verified, so Kaniuth's stronger version of Thoma's Theorem given in [14] shows that the left regular representation of $\Gamma_3 \rtimes_\vartheta \mathbb{Z}$ is not Type I. Since V is Type I, they cannot be equivalent. □

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