Operator Theory:
Advances and Applications, Vol. 41

C^*-ALGEBRAS OF CRYSTAL GROUPS

Keith F. Taylor

A explicit description is given for the C^*-algebra of a group with an abelian subgroup of finite index. An example is given to illustrate the ease of the construction in any particular case. Finally an extension of the main theorem to the case where the subgroup of finite index is not necessarily abelian is given.

Let G be a locally compact group with an abelian normal subgroup A of finite index n in G. The symmetry group of any crystal is of this type. In this paper the C^*-algebra, $C^*(G)$, of G is described in detail.

In general, the structure of group C^*-algebras have been difficult to understand with detailed descriptions given in only a few isolated cases, see [1] or [3], for examples. If G splits as a semidirect product of a finite group D with A, then $C^*(G)$ is isomorphic to the cross-product C^*-algebra of D acting on $C_0(\hat{A})$. In this case Rieffel has shown that D acts as automorphisms of $M_n(\mathbb{C}) \otimes C_0(\hat{A})$ and $C^*(G)$ is isomorphic to the fixed point algebra for this finite group of automorphisms. Actually Rieffel's result, 4.3 of [2] is more general in that $C_0(\hat{A})$ can be any C^*-algebra on which a finite group is acting.

Even when G is not a semidirect product of A by a finite group, let $D = G/A$. The main result here is that there is an injective homomorphism of D into the automorphism group of $M_n(\mathbb{C}) \otimes C_0(\hat{A})$ such that $C^*(G)$ is isomorphic to the fixed point algebra. Furthermore, the method of proof provides a very easy technique for giving an extremely explicit description of the C^*-algebra of any particular group with an abelian subgroup of finite index.

Using a detailed analysis of the Mackey procedure, Raeburn in [1] gave explicit descriptions of the C^*-algebras of two particular groups of the kind considered here. Raeburn's paper was an original motivation for this work and the procedure given below reproduces his descriptions when applied to the two examples he considered.

All the necessary terminology and the main result are given in section 2. An example of a non-symmorphic two dimensional crystal group is given in section 3 and its C^*-algebra is easily constructed using the main results. Section 4 is an addendum essentially due to the referee who noticed that the arguments of the main theorem, when properly viewed, applied to the case where the subgroup of finite index is not abelian.

2. DESCRIPTION OF $C^*(G)$

For a general locally compact group G, $C^*(G)$ is the completion of $L^1(G)$ with respect to the C^*-norm determined by the set of all *-representations of $L^1(G)$. If G is an amenable group then this C^*-norm is already determined by the left regular representation λ^G of $L^1(G)$ on $L^2(G)$. For $f \in L^1(G)$ and $h \in L^2(G)$, $\lambda^G_f h = f * h$, where $f * h(x) = \int_G f(y)h(y^{-1}x)dy$, for almost every x in G. Then λ^G is a *-representation of $L^1(G)$ on $L^2(G)$. The reduced C^*-algebra of G is $C^*_\lambda(G)$, the norm closure of $\lambda^G[L^1(G)]$ in $\mathcal{B}(L^2(G))$. If the group G is amenable, then $C^*(G)$ is isomorphic to $C^*_\lambda(G)$.

The abelian case will be used to establish some notation and illustrate the above definition. Of course, abelian groups are amenable. Let A be an abelian locally compact group with Pontryagin dual $\hat{A}$. Fix a Haar measure on A. For $f \in L^1(A)$, define the Fourier transform of f, $\hat{f}$ on $\hat{A}$, by $\hat{f}(\chi) = \int_A f(a)\chi(a)da$, for all $\chi \in \hat{A}$. Then $\hat{f} \in C_0(\hat{A})$ and the Stone-Weierstrass theorem can be used to show that $\{\hat{f}: f \in L^1(A)\}$ is dense in $(C_0(\hat{A}), \|\cdot\|_\infty)$. Fix a Haar measure on $\hat{A}$ so that, for any $f \in L^1(G) \cap L^2(G)$, $\|\hat{f}\|_2 = \|f\|_2$. Let $\mathcal{P}$ be the unitary map of $L^2(A)$ onto $L^2(\hat{A})$ such that $\mathcal{P}(h) = \hat{h}$ for any $h \in L^1(A) \cap L^2(A)$. For any $f \in L^1(A)$ and $h \in L^2(A)$, $\mathcal{P}(f * h) = \hat{f}\mathcal{P}(h)$.

Consider the representation $\mathcal{M}$ of $C_0(\hat{A})$ into $\mathcal{B}(L^2(\hat{A}))$ defined, for $f \in C_0(\hat{A})$ and $h \in L^2(\hat{A})$, by $\mathcal{M}_f h = fh$. Then $\mathcal{M}$ is a C^*-isomorphism of $C_0(\hat{A})$ with a C^*-subalgebra of $\mathcal{B}(L^2(\hat{A}))$. From the above discussion we have that $\mathcal{P}\lambda^A_f\mathcal{P}^{-1} = \mathcal{M}_{\hat{f}}$ for each $f \in L^1(A)$. Then $C^*(A)$ is isomorphic to $\mathcal{M}(C_0(\hat{A}))$, the closure of $\{\mathcal{P}\lambda^A_f\mathcal{P}^{-1}: f \in L^1(A)\}$ in $\mathcal{B}(L^2(\hat{A}))$ and $f \to \mathcal{M}^{-1}(\mathcal{P}\lambda^A_f\mathcal{P}^{-1})$ extends to a C^*-isomorphism of $C^*(A)$ with $C_0(\hat{A})$.

Now suppose G is a locally compact group with A as an abelian normal subgroup of finite index n in G. All the above notation will be retained for A. Let $D = G/A$. This group G is amenable, so $C^*(G) = C^*_\lambda(G)$. For any coset d of A, the set of elements of $L^2(G)$ supported on d is a subspace isometric with $L^2(A)$ and $L^2(G)$ is the orthogonal direct sum of these subspaces. Thus there is a natural isometry Ψ of $L^2(G)$ with $\mathcal{H} = \sum_{d\in D} \oplus L^2(\hat{A})$. Then $C^*(G)$ is isomorphic with the closure of $\{\Psi\lambda^A_f\Psi^{-1}: f \in L^1(G)\}$ in $\mathcal{B}(\mathcal{H})$. It is very useful to make this explicit.

Let $\gamma: D \to G$ be a fixed cross-section of the cosets of A in G. It is no loss of generality to assume γ takes the identity in D to the identity in G. By definition a cross-section satisfies $\gamma(d) \in d$ for each $d \in D$ and each d acts on A by $d \cdot a = \gamma(d)a\gamma(d)^{-1}$ for all $a \in A$. For $b, c \in D$, $\gamma(b)\gamma(c) \in bc$ and thus there exists $\alpha(b,c) \in A$ such that $\gamma(b)\gamma(c) = \gamma(bc)\alpha(b,c)$. Then α is the 2-cocycle which determines the isomorphism class of G given D, A and the action of D on A. Up to isomorphism $G = \{(d,a): d \in D,\ a \in A\}$, with group product $(b,a)(c,a') = (bc, \alpha(b,c)(c^{-1}\cdot a)a')$. The cocycle identity satisfied by α is vital in making the calculations below: $\alpha(b,cd)\alpha(c,d) = \alpha(bc,d)(d^{-1}\cdot\alpha(b,c))$ for all $b,c,d,\in D$.

To get an expression for the unitary Ψ mentioned above, for $h \in L^2(G)$ and each $d \in D$, let $h_d \in L^2(A)$ be such that $h_d(a) = h(d,a)$, for almost all a in A. The elements $\underline{h}$ of $\mathcal{H} = \sum_{d\in D} \oplus L^2(\hat{A})$ will be considered as n-tuples $\underline{h} = (h_d)_{d\in D}$, where each $h_d \in L^2(\hat{A})$. Then, for $h \in L^2(G)$, $\Psi(h) = \left(\mathcal{P}(h_d)\right)_{d\in D}$.

Let $M_n\left(C_0(\hat{A})\right)$ denote the algebra of $n\times n$-matrices over $C_0(\hat{A})$. It is convenient to index the matrix entries by elements of D. Thus $F \in M_n\left(C_0(\hat{A})\right)$ can be written $F = (F_{b,c})_{b,c\in D}$, with $F_{b,c} \in C_0(\hat{A})$ for each $b,c \in D$. If $M_n(\mathbb{C})$ is given the operator norm, then it is a C^*-algebra and $M_n\left(C_0(\hat{A})\right)$ is naturally identified with $C_0\left(\hat{A}, M_n(\mathbb{C})\right)$. Thus, $\|F\| = \sup_{x\in\hat{A}} \|F(x)\|$ defines a norm on $M_n\left(C_0(\hat{A})\right)$ with respect to which it is a C^*-algebra. It is also often convenient to realize $M_n\left(C_0(\hat{A})\right)$ as $M_n(\mathbb{C}) \otimes C_0(\hat{A})$.

For $F = (F_{b,c})_{b,c\in D}$ in $M_n\left(C_0(\hat{A})\right)$ define $\mathcal{M}(F)$ in $\mathcal{B}(\mathcal{H})$ by for $\underline{h} = (h_c)_{c\in D}$, $(\mathcal{M}(F)\underline{h})_b = \sum_{c\in D} F_{b,c}h_c$, for each $b \in D$. Then $\mathcal{M}$ is a C^*-isomorphism of $M_n\left(C_0(\hat{A})\right)$ into $\mathcal{B}(\mathcal{H})$.

Proposition 1. *For each $f \in L^1(G)$, $\Psi\lambda^G_f\Psi^{-1}$ is in the range of $\mathcal{M}$. Let $\mathcal{F}(f) = \mathcal{M}^{-1}(\Psi\lambda^G_f\Psi^{-1})$. Then $\mathcal{F}$ extends to a C^*-isomorphism of $C^*(G)$ onto a C^*-*

subalgebra of $M_n(C_0(\hat{A}))$.

Proof. The last statement of the proposition follows from the first and the facts that G is amenable, Ψ is a unitary and $\mathcal{M}$ is a C^*-isomorphism onto its range. The first statement is established by computing the effect of $\Psi\lambda_f^G$ on $h \in L^2(G)$, for $f \in L^1(G)$.

As before for $d \in D$, $f_d \in L^1(A)$ and $h_d \in L^2(G)$ are such that $f_d(a) = f(d,a)$ and $h_d(a) = h(d,a)$ for almost all $a \in A$. Then using the cocycle identity where required,

$$\begin{aligned}(f*h)_d(a') = f*h(d,a') &= \int_G f(y)h(y^{-1}(d,a'))dy \\ &= \sum_{c\in D}\int_A f(c,a)h((c,a)^{-1}(d,a'))da \\ &= \sum_{c\in D}\int_A f_c(a)h(c^{-1}d, \alpha(c,c^{-1}d)^{-1}(d^{-1}c\cdot a^{-1})a')da \\ &= \sum_{c\in D}\int_A f_c(c^{-1}d\cdot a)h_{c^{-1}d}(\alpha(c,c^{-1}d)^{-1}a^{-1}a')da \\ &= \sum_{c\in D}\int_A f_c(c^{-1}d\cdot(\alpha(c,c^{-1}d)^{-1}a))h_{c^{-1}d}(a^{-1}a')da \\ &= \sum_{c\in D} g_{c,d} * h_{c^{-1}d}(a')\ ,\end{aligned}$$

where $g_{c,d} \in L^1(A)$ is such that $g_{c,d}(a) = f_c(c^{-1}d\cdot(\alpha(c,c^{-1}d)^{-1}a))$ for almost all a in A. Therefore

$$\begin{aligned}(\Psi(f*h))_d &= \sum_{c\in D}\hat{g}_{c,d}\mathcal{P}(h_{c^{-1}d}) \\ &= \sum_{c\in D}\hat{g}_{dc^{-1},d}\mathcal{P}(h_c)\ .\end{aligned}$$

Since $\hat{g}_{dc^{-1},d} \in C_0(\hat{A})$, for each $c,d \in D$, it is now clear that $\Psi\lambda_f^G\Psi^{-1}$ is in the range of $\mathcal{M}$.

The action of D on A generates an action on $\hat{A}$. For $d \in D$ and $\chi \in \hat{A}$, let $d\cdot\chi(a) = \chi(d^{-1}\cdot a)$, for all $a \in A$.

Proposition 2. *For any* $f \in L^1(G)$, $\mathcal{F}(f)$ *is the element of* $M_n(C_0(\hat{A}))$ *whose entries are given by*

$$(\mathcal{F}(f))_{b,c}(\chi) = \chi(\alpha(bc^{-1},c))(\hat{f}_{bc^{-1}})(c\cdot\chi),$$

for all $\chi \in \hat{A},\ b,c \in D$.

Proof. The proof of proposition 1 shows that $(\mathcal{F}(f))_{b,c} = \hat{g}_{bc^{-1},b}$, where

$g_{c,d}(a) = f_c\big(c^{-1}d \cdot (\alpha(c, c^{-1}d)^{-1}a)\big)$. A direct computation shows that $\hat{g}_{c,d}(\chi) = \chi\big(\alpha(c, c^{-1}d)\big)\hat{f}_c(c^{-1}d \cdot \chi)$, for each $\chi \in \hat{A}$. A simple change of indices then establishes the formula in the proposition.

The next step is to characterize the range of $\mathcal{F}$ inside $M_n\big(C_0(\hat{A})\big)$.

Theorem 1. *There is an injective homomorphism of the finite group D into the automorphism group of $M_n\big(C_0(\hat{A})\big)$. The fixed point algebra $M_n\big(C_0(\hat{A})\big)^D$ under the resulting group of automorphisms is a C^*-subalgebra of $M_n\big(C_0(\hat{A})\big)$ and $\mathcal{F}$ extends from $L^1(G)$ to a C^*-isomorphism of $C^*(G)$ with $M_n\big(C_0(\hat{A})\big)^D$.*

Proof. Let ρ denote the right regular representation of G on $L^2(G)$ given by $\rho(y)h(x) = h(xy)$ for $h \in L^2(G)$, $x, y \in G$. Each $\rho(y)$ is a unitary operator on $L^2(G)$. Then, for any $y \in G$ and $f \in L^1(G)$, $\rho(y)\lambda^G_f\rho(y)^* = \lambda^G_f$. In particular, for $d \in D$, $\rho(\gamma(d))$ commutes with $\lambda^G\big(L^1(G)\big)$. Let $U(d) = \Psi\rho\big(\gamma(d)\big)\Psi^{-1}$, a unitary on $\mathcal{H}$ for each $d \in D$. For $\underline{h} = (h_c)_{c\in D}$ in $\mathcal{H}$, direct computations show that, for any $b \in D$ and $\chi \in \hat{A}$, $(U(d)\underline{h})_b(\chi) = \overline{d^{-1} \cdot \chi(\alpha(b,d))}h_{bd}(d^{-1} \cdot \chi)$, and $(U(d)^*\underline{h})_b(\chi) = \chi(\alpha(bd^{-1}, d))h_{bd^{-1}}(d \cdot \chi)$. For $F = (F_{b,c})_{b,c\in D}$ in $M_n\big(C_0(\hat{A})\big)$, $\big[U(d)\mathcal{M}(F)U(d)^*\underline{h}\big]_b(\chi) = \sum_{c\in D} d^{-1} \cdot \chi\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1} \cdot \chi)h_c(\chi)$. Thus conjugation by $U(d)$ leaves the range of $\mathcal{M}$ invariant and an automorphism $\beta(d)$ of $M_n\big(C_0(\hat{A})\big)$ can be defined by $\beta(d)(F) = \mathcal{M}^{-1}\big(U(d)\mathcal{M}(F)U(d)^*\big)$, for each $F \in M_n\big(C_0(\hat{A})\big)$. Then $\big[\beta(d)(F)\big]_{b,c}(\chi) = d^{-1} \cdot \chi\big(\alpha(b,d)^{-1}\alpha(c,d)\big)F_{bd,cd}(d^{-1} \cdot \chi)$ and further easy computations show that β is a homomorphism of D into the automorphism group of $M_n\big(C_0(\hat{A})\big)$. Since any F with only one entry nonzero has that entry moved by any d in D different then the identity, β is cleary injective.

Let $M_n\big(C_0(\hat{A})\big)^D = \{F \in M_n\big(C_0(\hat{A})\big)\colon\ \beta(d)(F) = F, \text{ for all } d \in D\}$. Then, for any $f \in L^1(G)$, $\rho(\gamma(d))\lambda^G_f\rho(\gamma(d))^* = \lambda^G_f$, implies that $\mathcal{F}(f) \in M_n\big(C_0(\hat{A})\big)^D$. The proof of theorem 1 will be completed by showing that $\mathcal{F}\big(L^1(G)\big)$ is dense in $M_n\big(C_0(\hat{A})\big)^D$. Let e denote the identity in D. For any $F \in M_n\big(C_0(\hat{A})\big)^D$, the e-column, $(F_{b,e})_{b\in D}$ and D-invariance completely determine F. For $f \in L^1(G)$, $(\mathcal{F}(f))_{b,e} = \hat{f}_b$, for each $b \in D$.

For fixed $F \in M_n\big(C_0(\hat{A})\big)^D$ and $\varepsilon > 0$, choose $\delta > 0$ so that, $F' \in M_n\big(C_0(\hat{A})\big)$ with $\|F'_{b,c} - F_{b,c}\|_\infty < \delta$ for all $b, c \in D$ implies $\|F' - F\| < \varepsilon$. For each $b \in D$, pick $f_b \in L^1(A)$ such that $\|\hat{f}_b - F_{b,e}\|_\infty < \delta$. Define $f \in L^1(G)$ by $f(b,a) = f_b(a)$, for all $b \in D$, $a \in A$. Then $\|\mathcal{F}f - F\| < \varepsilon$. Thus $\mathcal{F}(L^1(G))$ is dense in $M_n\big(C_0(\hat{A})\big)^D$.

Remark. $M_n\big(C_0(\hat{A})\big)^D$ is the algebra of matrices F whose e-column consists of n arbitrary $C_0(\hat{A})$ elements $(F_{d,e})_{d\in D}$ with every other entry determined by D-invariance. From the formulas in the above proof one easily shows that $F_{b,c}(\chi) = c^{-1} \cdot \chi\big(\alpha(b,c^{-1})^{-1}\alpha(c,c^{-1})\big)F_{bc^{-1},e}(c^{-1} \cdot \chi)$, for $\chi \in \hat{A}$, $b, c \in D$. Thus, $C^*(G)$, being isomorphic to this algebra, is given a very detailed description.

3. AN EXAMPLE

The description of sectin 2 will be illustrated with an example from the two-dimensional crystal groups. In this example $A \approx \mathbb{Z}^2$ and $D = \{1,-1\} \approx \mathbb{Z}_2$. Let $G = \{(1,n,m),\ (-1,n,m): n,m \in \mathbb{Z}\}$ with group products given by the following list, for $k,l,m,n \in \mathbb{Z}$,

$$(1,k,l)(1,n,m) = (1,k+n,l+m)$$

$$(1,k,l)(-1,n,m) = (-1,k+n,m-l)$$

$$(-1,k,l)(1,n,m) = (-1,k+n,l+m)$$

$$(-1,k,l)(-1,n,m) = (1,k+n+1,m-l)$$

with $A = \{(1,n,m): n,m \in \mathbb{Z}\}$, $\hat{A}$ is isomorphic to the torus $\mathbf{T}^2 = \{(z,w): z,w \in \mathbf{C},\ |z| = |w| = 1\}$. For $(z,w) \in \mathbf{T}^2$, let $\chi^{z,w} \in \hat{A}$ be defined by $\chi^{z,w}(1,n,m) = z^n w^m$, for all $(1,n,m) \in A$. The actions of D on A and $\hat{A}$ are given by $(-1)\cdot(1,n,m) = (1,n,-m)$ and $(-1)\cdot\chi^{z,w} = \chi^{z,\overline{w}}$. A cross-section γ from D into G is given by $\gamma(1) = (1,0,0)$ and $\gamma(-1) = (-1,0,0)$. Then the 2-cocycle α has only one nonidentity value, $\alpha(-1,-1) = (1,1,0)$.

For ease of notation, write functions on $\hat{A}$ as functions of $(z,w) \in \mathbf{T}^2$. The remark at the end of section 2 can now be used to see that $C^*(G)$ is isomorphic to the algebra of matrix-valued functions F on $\mathbb{T}^2$ of the form:

$$F(z,w) = \begin{pmatrix} F_{11}(z,w) & zF_{1-1}(z,\overline{w}) \\ F_{1-1}(z,w) & F_{11}(z,\overline{w}) \end{pmatrix},$$

for all $(z,w) \in \mathbf{T}^2$, where F_{11} and F_{1-1} are arbitrary continuous functions.

Let $\Omega = \{(z,w) \in \mathbf{T}^2: Im(w) \geq 0\}$. For each $z \in \mathbf{T}$, let $\mathcal{R}_z = \{\left(\begin{smallmatrix} a & zb \\ b & a \end{smallmatrix}\right): a,b \in \mathbf{C}\}$, a $*$-subalgebra of $M_2(\mathbf{C})$. Then the map which restricts F in $M_2\big(C(\mathbf{T}^2)\big) = C\big(\mathbf{T}^2, M_2(\mathbf{C})\big)$ to Ω provides an isomorphism of $C^*(G)$ with

$$\big\{F \in C(\Omega, M_2(\mathbf{C})): F(z,\pm 1) \in \mathcal{R}_z, \text{ for each } z \in \mathbf{T}\big\}.$$

4. AN ADDENDUM

The arguments used to prove theorm 1 apply in greater generality if one removes the Fourier transform converting $C^*(A)$ to $C_0(\hat{A})$. This is the observation of the referee, who was kind enough to work out the details. Before stating the theorem some notation is required.

Let G be a locally compact group with a normal subgroup N of finite index, n, in G. Let $D = G/N$ and let $\gamma : D \to G$ be a fixed cross-section of the cosets of N in G. Let $U : L^2(G) \to \sum_{d\in D} \oplus L^2(N)$ denote the isomorphism given by $(Uh)_d(t) = h(\gamma(d)t)$, for $t \in N$, $d \in D$, and $h \in L^2(G)$. Let ρ denote the right regular representation. As before, $C^*_\lambda(N)$ and $C^*_\lambda(G)$ denote the reduced C^*-algebras of N and G respectively. For any $y \in G$ and $f \in L^1(G)$, $\rho(y)\lambda^G(f)\rho(y^{-1}) = \lambda^G(f)$, so $\rho(y)$ commutes with $C^*_\lambda(G)$.

For $F = (f_{b,c})_{b,c\in D} \in M_n(L^1(N))$, define $\tilde{\lambda}(F)$ in $\mathcal{B}(\sum_{d\in D} \oplus L^2(N))$ by

$$\left(\tilde{\lambda}(F)\underline{h}\right)_d = \sum_{c\in D} \lambda^N(f_{d,c})h_c\ , \qquad \text{for } \underline{h} = (h_c)_{c\in D} \in \sum_{d\in D} \oplus L^2(N)\ .$$

Then $\tilde{\lambda}$ extends to an isomorphism, also denoted $\tilde{\lambda}$ of $M_n(C^*_\lambda(N))$ into $\mathcal{B}(\sum_{d\in D} \oplus L^2(N))$. For ease of notation, identify $M_n(C^*_\lambda(N))$ with its image under $\tilde{\lambda}$.

Theorem 2. (the referee) *There is an injective homomorphism of D into the automorphism group of $M_n(C^*_\lambda(N))$ such that $C^*_\lambda(G)$ is isomorphic with the fixed point algebra $M_n(C^*_\lambda(N))^D$.*

Proof. Most of the computational details will be left out since they are very similar to those in the proof of theorem 1.

It is easy to see that for $f \in L^1(G)$, $U\lambda^G(f)U^* \in M_n(C^*_\lambda(N))$ so $UC^*_\lambda(G)U^* \subseteq M_n(C^*_\lambda(N))$. For $y \in G$, define $\alpha(y)T = U\rho(y)U^*TU\rho(y^{-1})U^*$, for each $T \in \mathcal{B}(\sum_{d\in D} \oplus L^2(N))$. One checks, by computation, that $\alpha(y)$ maps $M_n(C^*_\lambda(N))$ onto itself. Consider α as a map of G into the automorphism group of $M_n(C^*_\lambda(N))$. Clearly α is a homomorphism and one can check that the kernel of α is N (because $\rho(t)$ commutes with $C^*_\lambda(N)$ for each $t \in N$). Thus, α defines an injective homomorphism, also denoted α, of D into $M_n(C^*_\lambda(N))$. Since $\rho(y)$ commutes with $C^*_\lambda(G)$, $UC^*_\lambda(G)U^*$ is in the fixed point algebra of $\alpha(D)$.

If $T = (T_{b,c})_{b,c\in D} \in M_n(C^*_\lambda(N))$ is invariant under $\alpha(D)$ and $\varepsilon > 0$, let $F = \left(\lambda^N(f_{b,c})\right)_{b,c\in D} \in M_n(\lambda^N(L^1(N)))$, be such that $\| T - F \| < \varepsilon$. Replacing F by

$\frac{1}{n}\sum_{d\in D}\alpha(d)F$, which is still in $M_n(\lambda^N(L^1(N)))$ one can assume F is $\alpha(D)$ invariant. Define $f \in L^1(G)$ by $f(\gamma(d)t) = f_{d,e}(t)$ for $d \in D$, $t \in N$, where e is the identity in D. Then computing carefully yields $U\lambda^G(f)U^* = F$. Thus $U\lambda^G(L^1(G))U^*$ is dense in the fixed point algebra which implies that $UC^*_\lambda(G)U^*$ is the fixed point algebra in $M_n(C^*_\lambda(N))$ under $\alpha(D)$.

REFERENCES

[1] Raeburn, Iain. On group C^*-algebras of bounded representation dimension. *Trans. Amer. Math. Soc.*, 262 (1982), 629-644.

[2] Rieffel, Marc A. Actions of finite groups on C^*-algebras. *Math. Scand.* 47 (1980), 157-176.

[3] Rosenberg, Jonathan. The C^*-algebras of some real and p-adic solvable groups. *Pac. J. Math.* 65 (1976), 175-192.

Department of Mathematics
Univeristy of Saskatchewan
Saskatoon, Saskatchewan S7N OWO
CANADA