CRM Proceedings and Lecture Notes 18, 99-105 (1999)

EXTENSIONS OF THE HEISENBERG GROUP AND WAVELET ANALYSIS IN THE PLANE

ECKART SCHULZ
School of Mathematics
Suranaree University of Technology
Nakhon Ratchasima, 30000
THAILAND

KEITH F. TAYLOR
Department of Mathematics and Statistics
University of Saskatchewan
106 Wiggins Road
Saskatoon, SK S7N 5E6

1 INTRODUCTION

In this lecture, we describe a class of groups formed as semidirect products with the positive real numbers acting on the Heisenberg group as dilations. All except one of the groups in this class have square-integrable representations and each of these square-integrable representations leads to continuous wavelet transforms of elements of $L^2(\mathbb{R}^2)$. We compute the reconstruction formulae explicitly in these cases. The techniques that we use are based on those developed in [1]. We begin by briefly presenting the continuous wavelet transform on $L^2(\mathbb{R})$ that arises from the affine group.

2 THE AFFINE WAVELET TRANSFORM

Let

$$\mathcal{H}_+^2 = \{g \in L^2(\mathbb{R}) : \text{ supp } \hat{g} \subseteq (0, \infty) \text{ a.e. } \}$$

and

$$\mathcal{H}_-^2 = \{g \in L^2(\mathbb{R}) : \text{ supp } \hat{g} \subseteq (-\infty, 0) \text{a.e. } \}.$$

Then, $L^2(\mathbb{R}) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$.

For $a > 0$ and $b \in \mathbb{R}$ and any $g \in L^2(\mathbb{R})$, let

$$[\rho(b,a)g](t) = \frac{1}{\sqrt{a}}\, g(\frac{t-b}{a}), \text{ for all } t \in \mathbb{R} .$$

The function $\rho(b,a)g$ is a translated and dilated copy of g.

An affine wavelet for $\mathcal{H}^2_+$ is any $\omega \in \mathcal{H}^2_+$ such that

$$\int_0^\infty \frac{|\hat{\omega}(\gamma)|^2}{\gamma}\, d\gamma = 1.$$

With such ω fixed, let $\omega_{b,a} = \rho(b,a)\omega$, for all $b \in \mathbb{R}, a > 0$. The continuous wavelet transform of $f \in \mathcal{H}^2_+$ using the wavelet ω is the function $T_\omega f$ of (b,a) given by

$$[T_\omega f](b,a) = \langle f, \omega_{b,a}\rangle = \int_{-\infty}^{\infty} f(t)\overline{\omega_{b,a}(t)}dt\ .$$

The reconstruction formula for f is

$$f = \int_0^\infty \int_{-\infty}^{\infty} [T_\omega f](b,a)\omega_{b,a}\frac{db\, da}{a^2},$$

with the integral on the right converging in the weak sense in the Hilbert space $\mathcal{H}^2_+$. Of course, a properly modified version of this holds on $\mathcal{H}^2_-$.

In terms of square-integrable group representations, the group here is the affine group $G_{\text{aff}} = \{(b,a) : b \in \mathbb{R}, a \in \mathbb{R}^+\}$ which has the group product of (b,a) with (b',a') given by $(b,a)(b',a') = (b+ab', aa')$. For each (b,a) in $G_{\text{aff}}, \rho(b,a)$ is a unitary operator on $L^2(\mathbb{R})$ and $(b,a) \to \rho(b,a)$ is a unitary representation of G_{aff}. The representation ρ is irreducible and square-integrable.

3 A GENERALIZATION

The group G_{aff} can be viewed as a semidirect product formed by the multiplicative group of positive real numbers acting on $\mathbb{R}$ by dilations. This situation was generalized in [1] to the following: Let H be a closed subgroup of $GL_n(\mathbb{R})$, the general linear group of invertible $n \times n$ real matricies. Consider $\mathbb{R}^n$ as the space of n-dimensional column vectors and let $h \in H$ act on $\mathbf{x} \in \mathbb{R}^n$ by $\mathbf{x} \to h\mathbf{x}$, with simple matrix multiplication. Let

$$G = \mathbb{R}^n \rtimes H = \{(\mathbf{x}, h) : \mathbf{x} \in \mathbb{R}^n, h \in H\}$$

with the product of $(\mathbf{x}, h)$ and $(\mathbf{y}, k)$ in G given by

$$(\mathbf{x}, h)(\mathbf{y}, k) = (\mathbf{x} + h\mathbf{y}, hk).$$

The dual group to $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ but we prefer to distinguish it by considering it as $\widehat{\mathbb{R}^n} = \{\gamma = (\gamma_1, \cdots, \gamma_n) : \gamma_j \in \mathbb{R}, 1 \leq j \leq n\}$, the space of n dimensional row vectors. Each $\gamma \in \widehat{\mathbb{R}^n}$ acts as a character on $\mathbb{R}^n$ by $(\gamma, \mathbf{x}) = e^{2\pi i \gamma \mathbf{x}}$, for all $\mathbf{x} \in \mathbb{R}^n$. The group H acts on $\widehat{\mathbb{R}^n}$ with the action written on the right. For $h \in H$ and $\gamma \in \widehat{\mathbb{R}^n}, \gamma \cdot h$ is defined by

$$(\gamma \cdot h, \mathbf{x}) = (\gamma, h\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Clearly, $\gamma \cdot h = \gamma h$, simple matrix multiplication, so we will simply write γh for $\gamma \cdot h$.

For $g \in L^1(\mathbb{R}^n)$, the Fourier transform of g is denoted $\hat{g}$ and is given by

$$\hat{g}(\gamma) = \int_{\mathbb{R}^n} g(\mathbf{x})(\gamma, \mathbf{x}) d\mathbf{x} .$$

If $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{g} \in L^2(\widehat{\mathbb{R}^n})$ and $||\hat{g}||_2 = ||g||_2$. Consequently, there exists a unitary map $\mathcal{P}$ of $L^2(\mathbb{R}^n)$ onto $L^2(\widehat{\mathbb{R}^n})$, called the Plancherel transform, such that $\mathcal{P}_g = \hat{g}$, for $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

There is a natural unitary representation ρ of $G = \mathbb{R}^n \rtimes H$ on $L^2(\mathbb{R}^n)$ given, for $(\mathbf{x}, h) \in G$ and $g \in L^2(\mathbb{R}^n)$, by

$$\rho(\mathbf{x}, h) g(\mathbf{y}) = \delta(h)^{-\frac{1}{2}} g(h^{-1}(\mathbf{y} - \mathbf{x})),$$

for all $\mathbf{y} \in \mathbb{R}^n$, where $\delta(h) = |\det(h)|$.

Let π denote the unitary representation of G that is equivalent to ρ via $\mathcal{P}$. That is, π is a representation on $L^2(\widehat{\mathbb{R}^n})$ such that

$$\pi(\mathbf{x}, h) = \mathcal{P} \rho(\mathbf{x}, h) \mathcal{P}^{-1}, \text{ for all } (\mathbf{x}, h) \in G.$$

A simple calculation (see Proposition 1, [1]) shows that, for $\xi \in L^2(\widehat{\mathbb{R}^n})$,

$$\pi(\mathbf{x}, h) \xi(\gamma) = \delta(h)^{\frac{1}{2}} (\gamma, \mathbf{x}) \xi(\gamma h), \text{ for all } \gamma \in \widehat{\mathbb{R}^n}.$$

Generally, ρ (or π) is not an irreducible representation of G. To see this, suppose U is a measureable H-invariant subset of $\widehat{\mathbb{R}^n}$. Let

$$L^2(U) = \{\xi \in L^2(\widehat{\mathbb{R}^n}) : \chi_U \xi = \xi\},$$

where χ_U is the characteristic function of U. Let

$$\mathcal{H}_U^2 = \{g \in L^2(\mathbb{R}^n) : \mathcal{P}g \in L^2(U)\}.$$

Then, it is easy to check that $L^2(U)$ is a π-invariant subspace of $L^2(\widehat{\mathbb{R}^n})$ and, so, $\mathcal{H}_U^2$ is a ρ-invariant subspace of $L^2(\mathbb{R}^n)$. If U is such that both U and $\widehat{\mathbb{R}^n}\setminus U$ have positive measure, then $L^2(U)$ and $\mathcal{H}_U^2$ are proper subspaces and π and ρ are reducible.

Now, suppose that there exists a $\gamma \in \widehat{\mathbb{R}^n}$ such that the H-orbit $\gamma H = \{\gamma h : h \in H\}$ is open in $\widehat{\mathbb{R}^n}$ and that the map $h \to \gamma h$ is a bijection of H onto γH. Let $U = \gamma H$ and let ρ_U denote the subrepresentation of ρ corresponding to the subspace $\mathcal{H}_U^2$ of $L^2(\mathbb{R}^n)$. In Theorem 1 of [1], it was shown that ρ_U is an irreducible square-integrable representation of G.

There is a beautiful theory of generalized orthogonality relations for square-integrable representations due to Duflo and Moore [2]. In the situation at hand, the relevant aspects are as follows (see Theorem 2 of [1] for full details):

Let Ψ denote the Radon-Nikodym derivative of the left Haar measure of H, transfered to the orbit U, with respect to the Lebesgue measure of $\widehat{\mathbb{R}^n}$, restricted to U. Define an operator K whose domain is dom $K = \{g \in \mathcal{H}_U^2 : \mathcal{P}g/\Psi \in L^2(U)\}$ and, for $g \in$ dom K, define

$$Kg = \mathcal{P}^{-1}(\mathcal{P}g/\Psi).$$

The operator K is (generally) unbounded, self-adjoint and positive. It is a generalization of the "dimension" of ρ_U. The operator $K^{-\frac{1}{2}}$ has domain $\{g \in \mathcal{H}_U^2 : \hat{g}\Psi^{\frac{1}{2}} \in L^2(U)\}$ and $K^{-\frac{1}{2}}g = \mathcal{P}^{-1}((\mathcal{P}g)\Psi^{\frac{1}{2}})$. For $f, g \in \mathcal{H}_U^2$, let $V_{g,f}(\mathbf{x}, h) = \langle f, \rho_U(\mathbf{x}, h)g\rangle$, for all $(\mathbf{x}, h) \in G$. If $g \in$ dom $K^{-\frac{1}{2}}$, then $V_{g,f} \in L^2(G)$, for all $f \in \mathcal{H}_U^2$ and, for $f_1, f_2 \in \mathcal{H}_U^2$,

$$\langle V_{g,f_1}, V_{g,f_2}\rangle_{L^2(G)} = \langle f_1, f_2\rangle_{\mathcal{H}_U^2} ||K^{-\frac{1}{2}}g||^2_{\mathcal{H}_U^2}.$$

Thus, if we fix $\omega \in \mathcal{H}_U^2$ such that $||\hat{\omega}\Psi^{\frac{1}{2}}||_{L^2(U)} = 1$, then $f \to V_{\omega,f}$ is an isometry of $\mathcal{H}_U^2$ into $L^2(G)$ and we get the weakly convergent reconstruction formula, for $f \in \mathcal{H}_U^2$,

$$f = \int_H \int_{\mathbf{R}^n} \langle f, \omega_{\mathbf{x},h}\rangle \omega_{\mathbf{x},h} \frac{d\mathbf{x}\, dh}{\delta(h)}, \qquad (1)$$

where $\omega_{\mathbf{x},h}(\mathbf{y}) = \rho(\mathbf{x},h)\omega(\mathbf{y})$, for all $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$ and $(\mathbf{x},h) \in G$.

For fixed ω, let $T_\omega f(\mathbf{x},h) = \langle f, \omega_{\mathbf{x},h}\rangle = \langle f, \rho(\mathbf{x},h)\omega\rangle$, for all $f \in \mathcal{H}^2_U$. Then T_ω, from $\mathcal{H}^2_U$ into $L^2(G)$, can be considered as a continuous wavelet transform. Any closed subgroup H of $GL_n(\mathbb{R})$, which is such that it gives rise to a free open orbit U in $\widehat{\mathbb{R}^n}$, gives us another form of continuous wavelet decompositions on $\mathbb{R}^n$. We will investigate a family of such decompositions on $\mathbb{R}^2$ that arise by adding dilations to the Heisenberg group.

4 EXTENSIONS OF THE HEISENBERG GROUP

Let $\mathbf{H}$ denote the three dimensional, simply connected, two step, nilpotent Lie group which we prefer to realize by three by three matrices as follows:

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

To compress notation a bit, denote

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

by $[x,y,z]$. Thus, the group product in $\mathbf{H}$ is given by

$$[x_1, y_1, z_1][x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2].$$

See Chapter 1 of [3] for a discussion of the Heisenberg groups - $\mathbf{H}$ is one realization of the 3-dimensional Heisenberg group - and their elegant representation theory.

We will say that "dilations of $\mathbf{H}$" means an action of $\mathbb{R}^+$ on $\mathbf{H}$ that is diagonal with respect to the $[x,y,z]$ parametrization. Since the action must respect the group structure of $\mathbf{H}$, a dilation is specified by two parameters α and β, say. For each $a \in \mathbb{R}^+$, the dilation by a is the map $\delta_{\alpha,\beta}(a) : \mathbf{H} \to \mathbf{H}$ given by

$$\delta_{\alpha,\beta}(a)[x,y,z] = [a^\alpha x, a^\beta y, a^{\alpha+\beta} z],$$

for all $[x,y,z] \in \mathbf{H}$. Let

$$G_{\alpha,\beta} = \mathbf{H} \rtimes_{\alpha,\beta} \mathbb{R}^+ = \{([x,y,z],a) : [x,y,z] \in \mathbf{H}, a \in \mathbb{R}^+\} \; .$$

The group product in $G_{\alpha,\beta}$ is given by

$$([x,y,z],a)([x',y',z'],a') = ([x + a^\alpha x', y + a^\beta y', z + a^{\alpha+\beta} z' + x a^\beta y'], aa') \; .$$

When we refer to an extension of the Heisenberg group by dilations, we mean a group which is isomorphic to $G_{\alpha,\beta}$, for some $(\alpha,\beta) \in \mathbb{R}^2 \backslash (0,0)$. The question naturally arises of whether the $G_{\alpha,\beta}$ are distinct and it turns out that they are not. Our first task is to find a convenient cross-section of the isomorphism classes of the collection of $G_{\alpha,\beta}$'s .

It is instructive to realize $G_{\alpha,\beta}$ as a group of 3×3 matrices. The map

$$([x,y,z],a) \rightarrow \begin{pmatrix} a^{\alpha+\beta} & a^\beta x & z \\ 0 & a^\beta & y \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism of $G_{\alpha,\beta}$ onto a closed subgroup of $GL_3(\mathbb{R})$. It is clear that each $G_{\alpha,\beta}$ is a four dimensional simply connected solvable Lie group. Let $\mathbf{g}_{\alpha,\beta}$ denote its Lie algebra. Since each $G_{\alpha,\beta}$ is simply connected, G_{α_1,β_1} is isomorphic to G_{α_2,β_2} if and only if $\mathbf{g}_{\alpha_1,\beta_1}$ is isomorphic to $\mathbf{g}_{\alpha_2,\beta_2}$.

Now $\{([x,0,0],1) : x \in \mathbb{R}\}, \{([0,y,0],1) : y \in \mathbb{R}\}, \{([0,0,z],1) : z \in \mathbb{R}\}$ and $\{([0,0,0],a) : a \in \mathbb{R}^+\}$ are four one-parameter subgroups of $G_{\alpha,\beta}$. Let X, Y, Z, and $A_{\alpha,\beta}$ denote the corresponding generators in $\mathbf{g}_{\alpha,\beta}$. With the above 3×3-matrix realization of $G_{\alpha,\beta}$, it is easy to realize these basic elements of $\mathbf{g}_{\alpha,\beta}$ by matrices. This gives

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$A_{\alpha,\beta} = \begin{pmatrix} \alpha+\beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \; .$$

So we can realize $\mathbf{g}_{\alpha,\beta}$ is the linear span of $\{X, Y, Z, A_{\alpha,\beta}\}$. Now it is clear that $\mathbf{g}_{\alpha,\beta} = \mathbf{g}_{\alpha/\beta,1}$, if $\beta \neq 0$ and $\mathbf{g}_{\alpha,0} = \mathbf{g}_{1,0}$, for all $(\alpha, \beta) \neq (0,0)$. Thus, we reparametrize. Let

$$A_p = \begin{pmatrix} p+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for $p \in \mathbb{R}$ and

$$A_\infty = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $\mathbf{g}_p = \langle X, Y, Z, A_p \rangle$, for $p \in \mathbb{R} \cup \{\infty\}$. Let G_p be the simply connected Lie group

$$exp(\mathbf{g}_p) = \left\{ \begin{pmatrix} a^{p+1} & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}, a \in \mathbb{R}^+ \right\}.$$

Each extension of the Heisenberg group is isomorphic to one of these G_p's; however, the G_p's are still not mutually nonisomorphic.

Theorem: *Let $p, q \in \mathbb{R} \cup \{\infty\}, p \neq q$. Then $\mathbf{g}_p$ is isomorphic to $\mathbf{g}_q$ if and only if $p = 1/q$. If $q \neq \infty$ and $p = 1/q$, then an isomorphism $\Phi : \mathbf{g}_p \to \mathbf{g}_q$ is given by $\Phi(X) = -Y, \Phi(Y) = X, \Phi(Z) = Z$, and $\Phi(A_p) = pA_q$. An isomorphism $\Phi : \mathbf{g}_0 \to \mathbf{g}_\infty$ is the same on X, Y, and Z but $\Phi(A_0) = A_\infty$.*

Proof: One directly checks that Φ as given provides an isomorphism in the indicated cases.

Conversely, suppose $p, q \in \mathbb{R}$ and $\Phi : \mathbf{g}_p \to \mathbf{g}_q$ is an isomorphism of Lie algebras. Now $\{A_p, X, Y, Z\}$ and $\{A_q, X, Y, Z\}$ are vector bases of $\mathbf{g}_p$ and $\mathbf{g}_q$, respectively. The only nontrivial Lie brackets among the basis elements of $\mathbf{g}_p$ are

$$[X, Y] = Z, [A_p, X] = pX, [A_p, Y] = Y, \text{ and } [A_p, Z] = (p+1)Z.$$

Thus, $[\mathbf{g}_p, \mathbf{g}_p] = \langle X, Y, Z \rangle$, if $p \neq 0$ and $[\mathbf{g}_0, \mathbf{g}_0] = \langle Y, Z \rangle$. Hence, $\mathbf{g}_0$ cannot be isomorphic to any $\mathbf{g}_p$ with $p \neq 0$ or ∞. So we may assume $p, q \in \mathbb{R} \backslash \{0\}$.

Furthermore, the centers of $[\mathfrak{g}_p, \mathfrak{g}_p]$ and $[\mathfrak{g}_q, \mathfrak{g}_q]$ are then both $\mathbb{R}Z$. Thus, an isomorphism $\Phi : \mathfrak{g}_p \to \mathfrak{g}_q$ must map $\langle X, Y, Z\rangle$ to $\langle X, Y, Z\rangle$ and $\mathbb{R}Z$ to $\mathbb{R}Z$. These observations mean that the matrix of Φ with respect to the two basis $\{A_p, X, Y, Z\}$ and $\{A_q, X, Y, Z\}$ has the form:

$$\begin{pmatrix} \varphi_{11} & 0 & 0 & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & 0 \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & \varphi_{44} \end{pmatrix}.$$

Since this matrix must be full rank, $\varphi_{11} \neq 0$ and $\varphi_{44} \neq 0$. The relation $[A_p, Z] = (p+1)Z$ implies $[\Phi A_p, \Phi Z] = (p+1)\Phi Z$. That is,

$$[\varphi_{11} A_q + \varphi_{21} X + \varphi_{31} Y + \varphi_{41} Z, \varphi_{44} Z] = (p+1)\varphi_{44} Z$$

which reduces to $(q+1)\varphi_{11}\varphi_{44} = (p+1)\varphi_{44}$. Since φ_{11} and φ_{44} are nonzero, either $q = -1$ and $p = -1$ or $\varphi_{11} = \frac{p+1}{q+1}$. In the case that either $p = -1$ or $q = -1$, the other is forced to be -1 also. Thus, we may assume $p \neq -1$ and $q \neq -1$ from now on. Thus,

$$\varphi_{11} = \frac{p+1}{q+1}. \tag{2}$$

Similarly, $[\Phi A_p, \Phi X] = p\Phi X$ implies

$$q\varphi_{11}\varphi_{22} = p\varphi_{22} \tag{3}$$

and

$$\varphi_{11}\varphi_{32} = p\varphi_{32}. \tag{4}$$

The relation $[\Phi A_p, \Phi Y] = \Phi Y$ implies that

$$q\varphi_{11}\varphi_{23} = \varphi_{23}. \tag{5}$$

and

$$\varphi_{11}\varphi_{33} = \varphi_{33}. \tag{6}$$

Finally, $[\Phi X, \Phi Y] = \Phi Z$ gives

$$\varphi_{22}\varphi_{33} - \varphi_{32}\varphi_{23} = \varphi_{44}. \tag{7}$$

Using (7), we see that $\varphi_{22}\varphi_{33}$ and $\varphi_{32}\varphi_{23}$ cannot both be 0. Now $\varphi_{22}\varphi_{33} \neq 0$ forces $\varphi_{11} = p/q$ from (3) and $\varphi_{11} = 1$ from (6). But then we would have

$p = q$, contrary to our assumptions.

Thus $\varphi_{32}\varphi_{23} \neq 0$. Then (4) implies $\varphi_{11} = p$ and (5) implies $\varphi_{11} = 1/q$. Hence, $p = 1/q$. Thus, if $p \neq q$ and $\mathbf{g}_p \cong \mathbf{g}_q$, then $p = 1/q$. $\square$

Corollary: *$\{G_p : -1 \leq p \leq 1\}$ is a complete cross-section of the isomorphism classes of $\{G_{\alpha,\beta} : (\alpha, \beta) \in \mathbb{R}^2 \backslash (0,0)\}$.*

When we refer to an extension of the Heisenburg group by dilations, from now on, we will mean one of the G_p for $-1 \leq p \leq 1$.

5 SQUARE-INTEGRABLE REPRESENTATIONS OF G_p

Recall that

$$G_p = \left\{ \begin{pmatrix} a^{p+1} & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}, a \in \mathbb{R}^+ \right\}.$$

Noting that

$$\begin{pmatrix} a^{p+1} & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^{p+1} & x & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

leads us to realizing G_p in the form $\mathbb{R}^2 \rtimes H_p$. These are the kinds of groups that were studied in section 3. Let

$$H_p = \left\{ \begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} : x \in \mathbb{R}, a \in \mathbb{R}^+ \right\},$$

which is a closed subgroup of $GL_2(\mathbb{R})$. Then

$$\mathbb{R}^2 \rtimes H_p = \left\{ \left(\begin{pmatrix} z \\ y \end{pmatrix}, \begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} \right) : \begin{pmatrix} z \\ y \end{pmatrix} \in \mathbb{R}^2, \begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} \in H_p \right\}$$

is isomorphic to G_p via the map

$$\left(\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} \right) \rightarrow \begin{pmatrix} a^{p+1} & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Following the analysis of section 3, we consider the action of H_p on $\widehat{\mathbb{R}^2} = \{(\gamma,\mu) : \gamma,\mu \in \mathbb{R}\}$. That is, for $(\gamma,\mu) \in \widehat{\mathbb{R}^2}$

$$(\gamma,\mu)\begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} = (a^{p+1}\gamma, a\mu + x\gamma) \tag{8}$$

for

$$\begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} \in H_p.$$

If $p = -1$, then there are three kinds of orbits in $\widehat{\mathbb{R}^2}$;

1. for $\gamma \neq 0$,
$$(\gamma,0)H_{-1} = \{(\gamma,\mu) : \mu \in \mathbb{R}\}$$
2. for $\gamma = 0$,
$$(0,1)H_{-1} = \{(0,\mu) : \mu > 0\}$$
and
$$(0,-1)H_{-1} = \{(0,\mu) : \mu < 0\}$$
3. finally, $\{(0,0)\}$ is an orbit itself.

Thus, there are no open H_{-1}-orbits in $\widehat{\mathbb{R}^2}$. In fact, G_{-1} has no square-integrable representations and does not fit under the general analysis developed in [1]. However, it is interesting that G_{-1} has been studied in detail, from the point of view of generating frames, in [4], where it is called the one dimensional UT group.

Therefore, from now on, assume $-1 < p \leq 1$. By (8), we see that there are two open H_p-orbits in $\widehat{\mathbb{R}^2}$. They are

$$U_+ = (1,0)H_p = \{(\gamma,\mu) : \gamma > 0\}$$

and

$$U_- = (-1,0)H_p = \{(\gamma,\mu) : \gamma < 0\}.$$

Since $U_+ \cup U_-$ is co-null in $\widehat{\mathbb{R}^2}, wehaveL^2(\widehat{\mathbb{R}^2}) = L^2(U_+) \oplus L^2(U_-)$ and $L^2(\mathbb{R}^2) = \mathcal{H}^2_{U_+} \oplus \mathcal{H}^2_{U_-}$.

Let us apply the theory of section 3 to determine the form that the continuous wavelet transform and reconstruction formula (1) takes in $\mathcal{H}^2_{U_+}$. We get a different analysis for each $p, -1 < p \leq 1$.

As a manifold, $G_p = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. Let us write (x, y, z, a) for the element

$$\begin{pmatrix} a^{p+1} & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix}$$

of G_p. Also, write the elements of $\mathbb{R}^2$ in the form (s,t) instead of as a column. Then, the square-integrable representation of G_p associated with U_+ is, for $g \in \mathcal{H}^2_{U_+}$ and $(x,y,z,a) \in G_p$,

$$\rho_{U_+}(x,y,z,a)g(s,t) = \frac{1}{a^{(p+2)/2}} g\Big(\frac{as - az - xt + xy}{a^{p+2}}, \frac{t-y}{a}\Big)$$

for all $(s,t) \in \mathbb{R}^2$.

To find the operator K, we need the Radon-Nikodym derivative Ψ. That is found as follows: Fix an element in U_+, say $(1,0)$. Let $\xi \in C_{00}(U_+)$. Then, using left Haar integration on H_p

$$\int_{H_p} \xi((1,0)h)dh = \int_0^\infty \int_{-\infty}^\infty \xi(a^{p+1}, x) \frac{dx\, da}{a^{p+1}}$$

$$= \int_0^\infty \int_{-\infty}^\infty \xi(\gamma, \mu) \frac{1}{(p+1)\gamma^{\left(\frac{2p+1}{p+1}\right)}}\, d\mu d\gamma.$$

Thus,

$$\Psi(\gamma,\mu) = \frac{1}{(p+1)\gamma^{\left(\frac{2p+1}{p+1}\right)}}, \text{ for } (\gamma,\mu) \in U_+ \ .$$

Note that when $p = 0, \Psi(\gamma,\mu) = \gamma^{-2}$ and, when $p = 1, \Psi(\gamma,\mu) = \frac{1}{2}\gamma^{-\frac{3}{2}}$.

Now, we fix any $\omega \in \mathcal{H}^2_{U_+}$ such that

$$(p+1)\int_0^\infty \int_{-\infty}^\infty |\hat{\omega}(\gamma,\mu)|^2 \gamma^{\left(\frac{2p+1}{p+1}\right)}\, d\mu d\gamma = 1. \qquad (9)$$

Let, for $(x,y,z,a) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$,

$$\omega_{(x,y,z,a)}(s,t) = \frac{1}{a^{\left(\frac{p+2}{2}\right)}} \omega\Big(\frac{as - az - xt + xy}{a^{p+2}}, \frac{t-y}{a}\Big) \qquad (10)$$

for all $(s,t) \in \mathbb{R}^2$. For $f \in \mathcal{H}^2_{U_+}$, define $T_\omega f$ on $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ by

$$T_\omega(x,y,z,a) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s,t)\overline{\omega_{(x,y,z,a)}(s,t)}\, dsdt. \tag{11}$$

Then the reconstruction formula (1) becomes,

$$f = \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_\omega f(x,y,z,a)\omega_{(x,y,z,a)}\, \frac{dxdydzda}{a^{2p+3}}, \tag{12}$$

weakly in $\mathcal{H}^2_{U_+}$.

We will leave the calculations for $\mathcal{H}^2_{U_-}$ to the interested reader.

6 Conclusions

We looked at those groups that arise by adding dilations to the Heisenberg group structure of $\mathbb{R}^3$, where we considered that the dilation must respect the group structure. This led us to the class of groups $G_p, -1 \leq p \leq 1$. Each G_p could be alternately realized in the form $\mathbb{R}^2 \rtimes H_p$ for an appropriate 2-dimensional subgroup H_p of $GL_2(\mathbb{R})$. Using the general theory of [1] for groups of this form, we found that G_p has square-integrable representations when $-1 < p \leq 1$.

For each $p, -1 < p \leq 1$, formulas (9), (10), (11), and (12) constitute a form of continuous wavelet analysis on $\mathbb{R}^2$ that was discovered via representation theory.

References

[1] Bernier, D and Taylor, K.F., Wavelets from Square-Integrable Representations, SIAM J. Math. Anal., 27 (1996), pp. 594-608.

[2] Duflo, M. and Moore, C. C., On the Regular Representation of a Nonunimodular Locally Compact Group, J. Funct. Anal., 21 (1976), pp. 209-243.

[3] Folland, G. B., Harmonic Analysis in Phase Space, Annals of Mathematics Studies, No. 122, Princeton University Press (1989), Princeton.

[4] Hogan, J.A. and Lakey, J.D., Extensions of the Heisenberg Group by Dilations and Frames, Applied and Computational Harmonic Analysis, 2 (1995), pp. 174-199.